Symmetric stochastic Petri nets

by Lindsay A. Prisgrove
Gerald S. Shedler

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1. Introduction

The stochastic Petri net (SPN) model is well suited to formal representation of concurrency, synchronization, and communication (cf. Ajmone Marsan, Conte, and Balbo [1], Dugan [2], Molloy [3, 4], Natkin [5], and Symons [6]). Such models have application in the performance evaluation of distributed computer systems. In this paper we focus on discrete event simulation methods for SPN models with special structure.

An SPN is specified by a finite set of *places* and a finite number of *transitions* along with a *normal input function*, an *inhibitor input function*, and an *output function* (each of which associates a set of places with a transition). A *marking* of an SPN is an assignment of zero or more *tokens* to the places in the net. A transition is *enabled* whenever there is at least one token in each of its normal input places and no tokens in any of its inhibitor input places; otherwise, it is *disabled*. A transition *fires* by removing one token (per place) from a random subset of its normal input places and depositing one token (per place) in a random subset of its output places. Such "random inputs" and "random outputs" are specified in terms of new marking probabilities as defined below and are needed for representation of distributed computer systems. The stochastic process $\{X(t): t \geq 0\}$, where $X(t)$ is the marking of the SPN at time t, is called the *marking process*.

Informally, an SPN is symmetric if there is a mapping of places onto places and transitions onto transitions which preserves sets of enabled transitions, new marking probabilities, sets of new transitions, and clock setting distributions. An important application of symmetric SPN models is in the representation of ring networks with equally spaced, identical ports; cf. Loucks, Hamacher, and Preiss [7]. Exploiting properties of a symmetric SPN and underlying regenerative process structure, we establish steady state estimation procedures based on independent, nonidentically distributed blocks of the marking process. We also establish estimation procedures for passage times in the symmetric SPN setting. The symmetry property considered in this paper is used to increase the statistical efficiency of SPN simulation.

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Although steady state estimation for an arbitrary SPN is a very difficult problem, Haas and Shedler [8] have provided estimation procedures for SPN models with a marking process that is a regenerative process in continuous time. To establish the regenerative property for the marking process of an SPN, it is necessary to show the existence of an infinite sequence of random time points at which the process probabilistically restarts. It is often clear that a marking process probabilistically restarts when a particular transition fires, leaving the system with a fixed marking. For specific models, however, it is nontrivial to determine conditions (distributional assumptions) under which this occurs infinitely often with probability one. Using recurrence theory (Haas and Shedler [9]) for generalized semi-Markov processes (König, Matthes, and Nawrotzki [10, 11], Matthes [12], Whitt [13]), conditions are given in [8] which ensure that the marking process of an SPN is a regenerative process in continuous time with finite expected time between regeneration points. This result leads to a steady state estimation procedure which does not exploit the special structure of a symmetric SPN.

Section 2 provides the formal definition of the marking process of an SPN given in [8] along with the definition of a symmetric SPN. Proposition 4 provides conditions which ensure that the marking process of a symmetric SPN is a regenerative process in continuous time and that the expected time between regeneration points is finite. Using a geometric trials recurrence criterion (Iglehart and Shedler [14, 15]), Proposition 4 postulates the existence of a transition, e^*, and a marking, s_0', such that transition e^* fires and the new marking is s_0' infinitely often with probability one. Conditions on the old clocks ensure that the marking process probabilistically restarts at these times. This result is the basis for regenerative simulation in the symmetric SPN setting.

Section 3 considers the steady state estimation problem for symmetric SPN models. The key observation is that under the assumptions of Proposition 4, regenerative cycles of the marking process defined by the times at which transition e^* fires and the new marking is s_0' can be decomposed into *independent, nonidentically distributed blocks*. The result of Proposition 6 and the ratio formula of Proposition 7 imply that in a symmetric SPN point estimates and confidence intervals for characteristics of symmetric functions of the limiting distribution can be based on short (independent, nonidentically distributed) blocks, rather than on long (independent, identically distributed) regenerative cycles of the marking process.

In Section 4 we establish estimation procedures for passage times in symmetric SPN models. Formal specification of a sequence $\{P_n': n \geq 1\}$ of passage times in a symmetric SPN with marking process $\{X(t): t \geq 0\}$, marking set, S, and transition set, E, is in terms of four subsets (A_1, A_2, B_1, and B_2) of S. The sets B_1 and B_2 define the random times $\{T_j': j \geq 1\}$ at which a passage time terminates. (The sets A_1 and A_2 define the random times at which a passage time starts.) Proposition 9 postulates the existence of $e^* \in E$ and $s_0, s_0' \in S$ such that transition e^* fires and the marking changes from s_0 to s_0' infinitely often with probability one: These transition firing times correspond to termination of a passage time with no other passage times underway. Conditions on the "old clocks" ensure that $\{(X(T_n'), P_{n+1}'): n \geq 0\}$ is a regenerative process in discrete time and that the expected time between regeneration points is finite.

Terminations of passage times that occur when no other passage times are underway and when transition e^* fires and the marking changes from s_0 to s_0' define regenerative cycles for the process $\{(X(T_n'), P_{n+1}'): n \geq 0\}$. The regenerative structure guarantees that $P_n' \Rightarrow P$ as $n \rightarrow \infty$ and the goal of the simulation is the estimation of $E\{f(P)\}$, where f is a real-valued measurable function. Estimates for $E\{f(P)\}$ can be based on simulation of the underlying marking process of the SPN in regenerative cycles. Alternatively, by exploiting properties of a symmetric SPN, these regenerative cycles can be decomposed into independent, nonidentically distributed blocks, and estimates for $E\{f(P)\}$ can be based on measurement of the passage times $\{P_n': n \geq 1\}$ contained in these blocks. This estimation procedure extracts more passage time information from a simulation of fixed length and provides estimates for $E\{f(P)\}$ that are relatively more accurate. In Section 5 we show that estimation based on these blocks indeed leads to shorter confidence intervals.

2. Regenerative stochastic Petri nets

Heuristically, an SPN changes marking in accordance with the firing of a transition enabled in the current marking. (We assume throughout that no two transitions fire simultaneously.) Each of the transitions enabled in a marking competes to change the marking, and each of these enabled transitions has its own stochastic mechanism for determining the next marking. When a transition in the SPN fires, new transitions may become enabled. For each of these new enabled transitions, a clock indicating the time until the transition fires is set according to an independent stochastic mechanism. (There is no restriction to exponentially distributed transition firing times.) If an enabled transition does not trigger a marking change but is enabled in the next marking, its clock continues to run; if such a transition is not enabled in the next marking, its clock reading is abandoned.

Following Haas and Shedler [8], formal definition of the marking process of an SPN is in terms of a general state space Markov chain (GSSMC) which describes the SPN at successive epochs of transition firing. Let $D = \{d_1, d_2, \cdots, d_L\}$ be a finite set of *places*, and let $E = \{e_1, e_2, \cdots, e_M\}$ be a finite set of *transitions*. Denote by S the countable set of *markings* and for $s \in S$ write $s = (s_1, s_2, \cdots, s_L)$, where s_j is the number of tokens in place $d_j \in D$. Denote the set of the *normal input places* for transition $e \in E$ by $I(e) \subseteq D$, the set

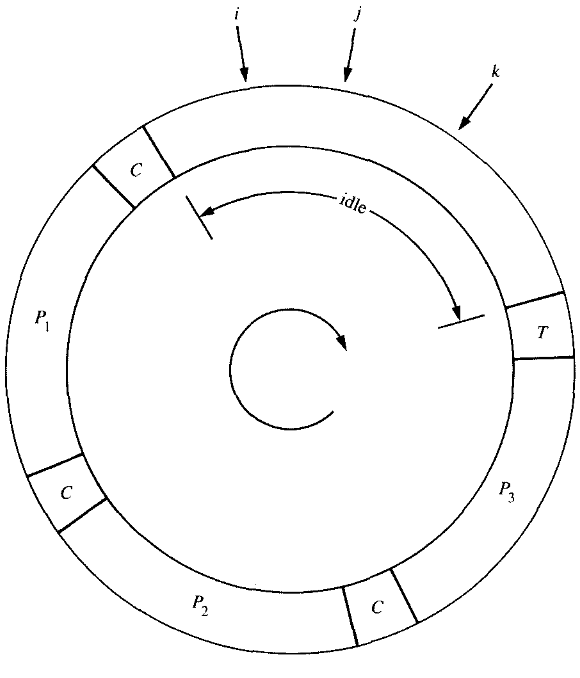

Figure 1

Token ring.

of the *inhibitor input places* by $L(e) \subseteq D$, and the set of the *output places* by $J(e) \subseteq D$. We assume that $L(e) \cap I(e) = \varnothing$ for all $e \in E$. For $s = (s_1, s_2, \cdots, s_L) \in S$, set

$$E(s) = \{e \in E: s_j \geq 1 \text{ for } d_j \in I(e) \text{ and } s_j = 0 \text{ for } d_j \in L(e)\}$$

so that $E(s)$ is the set of transitions that are enabled when the marking of the SPN is s. When the marking of the SPN is s, the firing of an enabled transition $e \in E(s)$ triggers a marking change to s'. We denote by $p(s'; s, e)$ the probability that the new marking is s', given that transition e fires when the marking is s. For all $s = (s_1, s_2, \cdots, s_L)$, $s' = (s_1', s_2', \cdots, s_L') \in S$, and $e \in E(s)$, we assume that $p(s'; s, e) > 0$ only if

(i) $s_j' = s_j - 1$ or s_j for all $d_j \in I(e) \cap (D - J(e))$;
(ii) $s_j' = s_j - 1$ or s_j or $s_j + 1$ for all $d_j \in I(e) \cap J(e)$;
(iii) $s_j' = s_j$ or $s_j + 1$ for all $d_j \in J(e) \cap (D - I(e))$; and
(iv) $s_j' = s_j$ for all $d_j \in (D - J(e) - I(e))$.

The actual enabled transition e which triggers a marking change when the marking is s depends on *clocks* associated with the enabled transitions and the *speeds* at which these clocks run. Each such clock records the remaining time until the transition fires. We denote by r_{si} (≥ 0) the deterministic rate at which the clock associated with transition e_i runs when the marking is s; for each $s \in S$, $r_{si} = 0$ if $e_i \notin E(s)$. We assume that $r_{si} > 0$ for some $e_i \in E(s)$. (Typically in applications, all speeds r_{si} are equal to one. There are, however, models in which speeds other than unity as well as state-dependent speeds are convenient.)

For $s \in S$ define $C(s)$ to be the set of possible clock readings when the marking is s:

$$C(s) = \{(c_1, \cdots, c_M): c_i \geq 0 \text{ and } c_i > 0 \text{ if and only if } e_i \in E(s);\ c_i r_{si}^{-1} \neq c_j r_{sj}^{-1} \text{ for } i \neq j \text{ with } c_i c_j r_{si} r_{sj} > 0\}. \quad (1)$$

The conditions in Equation (1) ensure that no two transitions fire simultaneously, as defined below. The clock with reading c_i is said to be *active* when the marking is s if transition e_i is enabled $[e_i \in E(s)]$. For $s \in S$ and $c \in C(s)$, let

$$t^* = t^*(s, c) = \min_{\{i: e_i \in E(s)\}} \{c_i r_{si}^{-1}\}, \quad (2)$$

where $c_i r_{si}^{-1}$ is taken to be $+\infty$ when $r_{si} = 0$. Also set

$$c_i^* = c_i^*(s, c) = c_i - t^*(s, c) r_{si}, \quad (3)$$

for $e_i \in E(s)$ and

$$i^* = i^*(s, c) = i \text{ such that } e_i \in E(s) \text{ and } c_i^*(s, c) = 0. \quad (4)$$

Beginning with marking s and clock vector c, $t^*(s, c)$ is the time to the next transition firing and $i^*(s, c)$ is the index of the unique firing transition $e^* = e^*(s, c) = e_{i^*(s,c)}$.

At a marking change from s to s' triggered by the firing of transition e^*, new clock times are generated for each event $e' \in N(s'; s, e^*) = E(s') - (E(s) - \{e^*\})$. The distribution function of such a new clock time is denoted by $F(\cdot; s', e', s, e^*)$, and we assume that $F(0; s', e', s, e^*) = 0$. For $e' \in O(s'; s, e^*) = E(s') \cap (E(s) - \{e^*\})$, the old clock reading is kept after e^* fires. For $e' \in (E(s) - \{e^*\}) - E(s')$, transition e' (which was enabled before transition e^* fired) is disabled.

Next consider a GSSMC $\{(S_n, C_n): n \geq 0\}$ having state space

$$\Sigma = \bigcup_{s \in S} (\{s\} \times C(s))$$

and representing the marking (S_n) and vector (C_n) of clock readings at successive transition firing times. (The ith coordinate of the vector C_n is denoted by $C_{n,i}$.) The transition kernel of the Markov chain $\{(S_n, C_n): n \geq 0\}$ is

$$P((s, c), A) = p(s'; s, e^*) \prod_{e_i \in N(s')} F(a_i; s', e_i, s, e^*) \prod_{e_i \in O(s')} 1_{[0, a_i]}(c_i^*), \quad (5)$$

where $N(s') = N(s'; s, e^*)$, $O(s') = O(s'; s, e^*)$, and

$$A = \{s'\} \times \{(c_1', \cdots, c_M') \in C(s'): c_i' \leq a_i \text{ for } e_i \in E(s')\}.$$

The set A is the subset of Σ which corresponds to the SPN changing marking to s' with the reading c'_i on the clock associated with transition $e_i \in E(s')$ set to a value in $[0, a_i]$. [We suppose that the clock setting distributions are such that $P((s, c), \Sigma) = 1$ for all $(s, c) \in \Sigma$.]

Finally, the marking process of the SPN is a piecewise constant continuous time process constructed from the GSSMC $\{(S_n, C_n): n \geq 0\}$ in the following manner. Set $\zeta_0 = 0$ and

$$\zeta_n = \sum_{k=0}^{n-1} t^*(S_k, C_k),$$

$n \geq 1$. According to this definition, ζ_n is the nth time at which the marking process makes a state transition. [We assume that

$$P\{\sup_{n \geq 1} \zeta_n = +\infty \mid (S_0, C_0)\} = 1 \text{ a.s.}$$

for all initial states (S_0, C_0).] Then set

$$X(t) = S_{N(t)}, \tag{6}$$

where

$$N(t) = \max\{n \geq 0: \zeta_n \leq t\}. \tag{7}$$

The marking process of the SPN is the process $\{X(t): t \geq 0\}$ defined by Equation (6). Henceforth we restrict attention to SPN models in which all speeds r_{si} are equal to 1.

For ring networks with N ports, reference to port index j is to be interpreted as reference to index $j - 1 \pmod N + 1$. In the graphical representation of an SPN, places are drawn as circles and transitions as bars. Directed arcs connect transitions to output places and input places to transitions. Tokens are drawn as black dots.

Example 1 (*token ring*)
Consider a unidirectional ring network having a fixed number of *ports*, labeled $1, 2, \cdots, N$ in the direction of signal propagation; see **Figure 1.** At each port *message packets* arrive according to a random process. A single *control token* (denoted by T in Figure 1) circulates around the ring from one port to the next. The time for the ring token to propagate from port $j - 1$ to port j is a positive constant, R_{j-1}, $j = 1, 2, \cdots, N$. When a port observes the ring token and there is a packet queued for transmission, the port converts the token to a *connector* (C) and transmits a packet followed by the token pattern; the token continues to propagate if there is no packet queued for transmission. By destroying the connector prefix, the port removes the transmitted packet when it returns around the ring.

Assume that the time for port j to transmit a packet is a positive random variable, L_j, with finite mean. Also assume that packets arrive at individual ports randomly and independently of each other: The time from end of transmission by port j until the arrival of the next packet for transmission by port j is a positive random variable, A_j, with finite mean. Note that there is at most one packet queued for transmission at any time at any particular port.

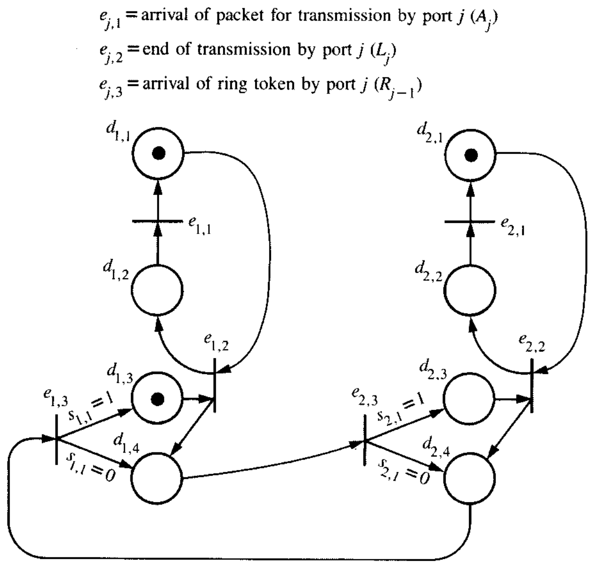

Figure 2

SPN representation of two-port token ring.

Following [8], let $D = \{d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, \cdots, d_{N,1}, d_{N,2}, d_{N,3}, d_{N,4}\}$ be the set of places of the SPN and let $E = \{e_{1,1}, e_{1,2}, e_{1,3}, \cdots, e_{N,1}, e_{N,2}, e_{N,3}\}$ be the set of transitions. See **Figure 2** for $N = 2$. (For clarity of exposition, we use double subscripts in the SPN representation of the token ring model to index places, transitions, and token counts.) Set

$$L(e_{j,1}) = L(e_{j,2}) = L(e_{j,3}) = \varnothing, \tag{8}$$

$$I(e_{j,1}) = \{d_{j,2}\},\ I(e_{j,2}) = \{d_{j,1}, d_{j,3}\},\ I(e_{j,3}) = \{d_{j-1,4}\}, \tag{9}$$

and

$$J(e_{j,1}) = \{d_{j,1}\},\ J(e_{j,2}) = \{d_{j,2}, d_{j,4}\},\ J(e_{j,3}) = \{d_{j,3}, d_{j,4}\}, \tag{10}$$

$j = 1, 2, \cdots, N$.

The transitions have the following interpretation: $e_{j,1}$ = "arrival of packet for transmission by port j," $e_{j,2}$ = "end of transmission by port j," and $e_{j,3}$ = "observation of ring token by port j," $j = 1, 2, \cdots, N$. The interpretation of the places is as follows. Let $t \geq 0$ and suppose that the marking of the SPN at time t is $s = (s_{1,1}, s_{1,2}, \cdots, s_{N,4})$. Then $s_{j,1} = 1$ if and only if at time t port j is transmitting a packet or there is a packet waiting for transmission by port j; $s_{j,2} = 1$ if and only if at time t port j is not transmitting a packet and there is no packet waiting for transmission by port j; $s_{j,3} = 1$ if and only if at time t port j is transmitting a packet; and $s_{j,4} = 1$ if and only if at time t the ring token is propagating to port $j + 1$, $j = 1, 2, \cdots, N$. (Otherwise $s_{j,k} = 0$.)

Set

$$S' = \{(s_{1,1}, s_{1,2}, \cdots, s_{N,4}): s_{j,1}, \cdots, s_{j,4} = 0 \text{ or } 1 \text{ for } 1 \le j \le N;$$
$$s_{1,1} + s_{1,2} + \cdots + s_{N,4} = N + 1\}.$$

Then the set, S, of markings is

$$S = \{(s_{1,1}, s_{1,2}, \cdots, s_{N,4}) \in S': s_{j,1} + s_{j,2} = 1 \text{ and}$$
$$s_{j,2}s_{j,3} = 0 \text{ for } 1 \le j \le N\}. \tag{11}$$

(It follows that $|S| = 3N2^{N-1}$. In any marking there are exactly $N + 1$ tokens. There is at most one token in each place. Each of the disjoint sets of places $\{d_{j,1}, d_{j,2}\}$ contains exactly one token indicating whether or not port j has a packet queued for transmission. The set of places $\{d_{1,3}, d_{1,4}, d_{2,3}, d_{2,4}, \cdots, d_{N,3}, d_{N,4}\}$ contains exactly one token indicating the position and status of the ring token. There can never be tokens at places $d_{j,2}$ and $d_{j,3}$ simultaneously, reflecting the fact that there can be no arrival of a packet for transmission by port j during a transmission by port j.)

The new marking probabilities $p(s'; s, e)$ are as follows. If $e = e_{j,1}$ = "arrival of packet for transmission by port j," then $p(s'; s, e) = 1$ when

$$s = (s_{1,1}, \cdots, s_{j-1,4}, 0, 1, 0, s_{j,4}, s_{j+1,1}, \cdots, s_{N,4}) \in S \text{ and}$$
$$s' = (s_{1,1}, \cdots, s_{j-1,4}, 1, 0, 0, s_{j,4}, s_{j+1,1}, \cdots, s_{N,4}).$$

If $e = e_{j,2}$ = "end of transmission by port j," then $p(s'; s, e) = 1$ when

$$s = (s_{1,1}, \cdots, s_{j-1,4}, 1, 0, 1, 0, s_{j+1,1}, \cdots, s_{N,4}) \in S \text{ and}$$
$$s' = (s_{1,1}, \cdots, s_{j-1,4}, 0, 1, 0, 1, s_{j+1,1}, \cdots, s_{N,4}).$$

If $e = e_{j,3}$ = "observation of ring token by port j," then $p(s'; s, e) = 1$ when

$$s = (s_{1,1}, \cdots, s_{j-1,3}, 1, 1, 0, 0, 0, s_{j+1,1}, \cdots, s_{N,4}) \in S \text{ and}$$
$$s' = (s_{1,1}, \cdots, s_{j-1,3}, 0, 1, 0, 1, 0, s_{j+1,1}, \cdots, s_{N,4})$$

and when

$$s = (s_{1,1}, \cdots, s_{j-1,3}, 1, 0, 1, 0, 0, s_{j+1,1}, \cdots, s_{N,4}) \in S \text{ and}$$
$$s' = (s_{1,1}, \cdots, s_{j-1,3}, 0, 0, 1, 0, 1, s_{j+1,1}, \cdots, s_{N,4}).$$

All other new marking probabilities $p(s'; s, e)$ are equal to zero.

Note that when transition $e_{j,3}$ fires, a token is removed from place $d_{j-1,4}$ and a token is deposited either in place $d_{j,3}$ or in place $d_{j,4}$, depending upon whether $(s_{j,1}, s_{j,2})$ equals $(1, 0)$ or $(0, 1)$. All other transitions are *input-deterministic* (in that exactly one token is removed from each input place when the transition fires) and *output-deterministic* (exactly one token is deposited in each output place when the transition fires).

The distribution functions of new clock times for transitions $e' \in N(s'; s, e^*)$ are as follows. If $e' = e_{j,1}$ = "arrival of packet for transmission by port j," then the distribution function $F(x; s', e', s, e^*) = P\{A_j \le x\}$. If $e' = e_{j,2}$ = "end of transmission by port j," then the distribution function $F(x; s', e', s, e^*) = P\{L_j \le x\}$. If $e' = e_{j,3}$ = "observation of ring token by port j," then the distribution function $F(x; s', e', s, e^*) = 1_{[R_{j-1},\infty)}(x)$.

We now define a symmetric SPN. Informally, an SPN is symmetric if there is a mapping of places onto places and transitions onto transitions which preserves the sets $E(s)$ of enabled transitions, the new marking probabilities $p(s'; s, e)$, the sets $N(s'; s, e^*)$ of new transitions, and the clock setting distributions $F(\cdot; s', e', s, e^*)$. Let $\{X(t): t \ge 0\}$ be the marking process of an SPN with finite marking set, S, and transition set, E. As before, we let $D = \{d_1, d_2, \cdots, d_L\}$ be the set of places and $E = \{e_1, e_2, \cdots, e_M\}$ be the set of transitions. We assume throughout this section that $L = L_1N$ and $M = M_1N$ for some $N \ge 2$. (We also assume that all clock setting distributions have finite mean.)

Let ϕ be a cyclic permutation of the set $\{1, 2, \cdots, N\}$. In terms of this permutation define a mapping, ϕ_D, of D onto D:

$$\phi_D(d_{(j-1)L_1+k}) = d_{(\phi(j)-1)L_1+k}, \tag{12}$$

$j = 1, 2, \cdots, N$ and $k = 1, 2, \cdots, L_1$. Similarly, define a mapping, ϕ_E, of E onto E:

$$\phi_E(e_{(j-1)M_1+k}) = e_{(\phi(j)-1)M_1+k}, \tag{13}$$

$j = 1, 2, \cdots, N$ and $k = 1, 2, \cdots, M_1$. Also define a mapping, ϕ_S, of S onto S:

$$\phi_S(s_1, s_2, \cdots, s_L) = (s_{\phi_D(1)}, s_{\phi_D(2)}, \cdots, s_{\phi_D(L)}). \tag{14}$$

For an arbitrary subset D' of D we write $\phi_D(D') = \{\phi_D(i): i \in D'\}$ and for an arbitrary subset E' of E we write $\phi_E(E') = \{\phi_E(e): e \in E'\}$. This notation facilitates formal definition of a symmetric SPN.

Definition 2
An SPN with marking process $\{X(t): t \ge 0\}$ is said to be *symmetric* if there exists a cyclic permutation, ϕ, of the set $\{1, 2, \cdots, N\}$ such that

(i) $\phi_D(L(e)) = L(\phi_E(e))$, $\phi_D(I(e)) = I(\phi_E(e))$, and $\phi_D(J(e)) = J(\phi_E(e))$ for all $e \in E$;
(ii) $p(s'; s, e) = p(\phi_S(s'); \phi_S(s), \phi_E(e))$ for all $e \in E$ and $s, s' \in S$; and
(iii) $F(\cdot; s', e', s, e) = F(\cdot; \phi_S(s'), \phi_E(e'), \phi_S(s), \phi_E(e))$ for all $e' \in N(s'; s, e)$, $e \in E$, and $s, s' \in S$.

Condition (i) ensures that the induced mappings ϕ_D and ϕ_E preserve the sets of normal input places, inhibitor input places, and output places, and thus [using Equation (14)] $\phi_E(E(s)) = E(\phi_S(s))$ for all $s \in S$. Conditions (ii) and (iii) ensure that the mappings ϕ_S and ϕ_E preserve the new marking probabilities and the clock setting distributions.

Example 3
In the token ring model of Example 1, let N be the number of ports so that $L_1 = 4$ and $M_1 = 3$. Suppose that (i) $R_1 = R_2 = \cdots = R_N$; (ii) the random variables $A_1, A_2, \cdots, A_N$ are identically distributed; and (iii) the random variables $L_1, L_2, \cdots, L_N$ are identically distributed. Under these assumptions the SPN is symmetric. For example, take $\phi(j) = j + 1$, $j = 1, 2, \cdots, N$. Then

$$\phi_D(d_{j,k}) = d_{\phi(j),k},$$

$$\phi_E(e_{j,k}) = e_{\phi(j),k},$$

and

$$\phi_S(s_{1,1}, s_{1,2}, s_{1,3}, s_{1,4}, \cdots, s_{N,4}) = (s_{2,1}, s_{2,2}, s_{2,3}, s_{2,4}, \cdots, s_{N,4}, s_{1,1}, s_{1,2}, s_{1,3}, s_{1,4}).$$

Checking condition (i) of Definition 2 for transition $e_{1,1}$, we have that

$$\phi_D(L(e_{1,1})) = \varnothing = L(e_{2,1}) = L(\phi_E(e_{1,1})),$$

$$\phi_D(I(e_{1,1})) = \phi_D(\{d_{1,2}\}) = \{d_{2,2}\} = I(e_{2,1}) = I(\phi_E(e_{1,1})),$$

and

$$\phi_D(J(e_{1,1})) = \phi_D(\{d_{1,1}\}) = \{d_{2,1}\} = J(e_{2,1}) = J(\phi_E(e_{1,1})).$$

These equations imply that $\phi_E(E(s)) = E(\phi_S(s))$ for all $s \in S$. For example, let $s = (0, 1, 0, 1, 0, 0, 0, 0)$. Then $\phi_S(s) = (0, 0, 0, 0, 0, 1, 0, 1)$ so that

$$E(\phi_S(s)) = \{e_{2,1}, e_{1,3}\} = \{\phi_E(e_{1,1}), \phi_E(e_{2,3})\} = \phi_E(E(s)).$$

To illustrate that condition (ii) is satisfied, let $e = e_{1,2}$ = "end of transmission by port 1," $s = (1, 0, 1, 0, 0, 0, 0, 0)$, and $s' = (0, 1, 0, 1, 0, 0, 0, 0)$. Then $\phi_E(e) = e_{2,2}$ = "end of transmission by port 2," $\phi_S(s) = (0, 0, 0, 0, 1, 0, 1, 0)$, and $\phi_S(s') = (0, 0, 0, 0, 0, 1, 0, 1)$ so that

$$p(s'; s, e) = 1 = p(\phi_S(s'); \phi_S(s), \phi_E(e)).$$

Next, let $e' = e_{1,1} = N(s'; s, e)$, and recall that the random variables A_1 and A_2 are identically distributed so that

$$F(x; s', e', s, e) = P\{A_1 \le x\} = P\{A_2 \le x\} = F(x; \phi_S(s'), \phi_E(e'), \phi_S(s), \phi_E(e)).$$

Conditions (i), (ii), and (iii) can be verified for all $e \in E$ and $s, s' \in S$ in a similar manner.

Proposition 4 gives a set of conditions on the building blocks of a symmetric SPN which ensure that the marking process is a regenerative process in continuous time and that the expected time between regeneration points is finite. This result is a direct consequence of Proposition 4.7 of [8].

Set $\phi^1_S(s) = \phi_S(s)$ for all $s \in S$ and $\phi^1_E(e) = \phi_E(e)$ for $e \in E$. Recall that ζ_n is the nth time at which the marking process makes a state transition, $n \ge 0$. Let $\{T^1_n: n \ge 0\}$ be an increasing sequence of stopping times that are finite ($T^1_n < \infty$ a.s.) transition firing times such that for some $e^* \in E$ and $S^* \subseteq S$, $T^1_0 = 0$ and

$$T^1_n = \inf\{t > T^1_{n-1}: \text{at time } t \text{ transition } \phi^1_E(e^*) \text{ fires and the marking is } \phi^1_S(s^*) \text{ for some } s^* \in S^*\}, \qquad n \ge 1.$$

Proposition 4
Suppose that there exists $s'_0 \in S$ and $\delta > 0$ such that

$$P\{X(T^1_n) = \phi^1_S(s'_0) \mid X(T^1_{n-1}), \cdots, X(T^1_1)\} \ge \delta \text{ a.s.} \tag{15}$$

Also suppose that there exists $s \in S$ such that for all $s^* \in S^*$,

(i) the set $O(s'_0; s^*, e^*) = E(s'_0) \cap (E(s^*) - \{e^*\}) = \varnothing$;
(ii) the set $N(s'_0; s^*, e^*) = E(s'_0) - (E(s^*) - \{e^*\}) = N(s'_0; s, e^*)$; and
(iii) the clock setting distribution $F(\cdot; s'_0, e', s^*, e^*) = F(\cdot; s'_0, e', s, e^*)$ for all $e' \in N(s'_0; s, e^*)$.

Then $\{X(t): t \ge 0\}$ is a regenerative process in continuous time. Moreover, if

$$E\{T^1_{n+1} - T^1_n\} \le c < \infty \tag{16}$$

for all $n \ge 0$, then the expected time between regeneration points is finite.

Equation (15) implies that transition $\phi^1_E(e^*)$ triggers a marking change to $\phi^1_S(s'_0)$ infinitely often with probability one. Furthermore, at such a time T^1_n, the only clocks that are active have just been set, since $O(\phi^1_S(s'_0); \phi^1_S(s^*), \phi^1_E(e^*)) = \varnothing$ for all $s^* \in S^*$. The joint distribution of $X(T^1_n)$ and the clocks set at time T^1_n depends on the past history of $\{X(t): t \ge 0\}$ only through $\phi^1_S(s'_0)$, the previous marking $\phi^1_S(s^*)$, and the trigger transition $\phi^1_E(e^*)$. Since the new transitions and clock setting distributions are the same for all s^*, the process $\{X(t): t \ge 0\}$ probabilistically restarts whenever $\{X(T^1_n): n \ge 1\}$ hits $\phi^1_S(s'_0)$. Note that the result of Proposition 4 also holds if condition (i) is replaced by (i'): $O(s'_0; s_0, e^*) \ne \varnothing$, and for any $e' \in O(s'_0; s_0, e^*)$ the clock setting distribution $F(\cdot; s', e', s, e)$ is exponential with mean which is independent of s, s', and e. [Assumption (i') ensures that no matter when the clock for transition $e' \in O(s'_0; s_0, e^*)$ was set, the remaining time until transition e' triggers a marking change is exponentially distributed with the same mean.]

Under the conditions of Proposition 4, the basic limit theorem for regenerative processes asserts that $X(t) \Rightarrow X$ as $t \to \infty$. The goal of the simulation is the estimation of $r(f) = E\{f(X)\}$, where f is a real-valued (measurable) function having domain S. From n cycles the standard regenerative method (Crane and Iglehart [16]) provides the strongly consistent point estimate

$$\hat{r}(n) = \frac{\bar{Y}(n)}{\bar{\tau}(n)} \tag{17}$$

and asymptotic $100(1 - 2\gamma)\%$ confidence interval

$$\left[\hat{r}(n) - \frac{z_{1-\gamma}s(n)}{\bar{\tau}(n)n^{1/2}},\ \hat{r}(n) + \frac{z_{1-\gamma}s(n)}{\bar{\tau}(n)n^{1/2}}\right] \tag{18}$$

for $r(f)$. In Equation (17)

$$\bar{Y}(n) = n^{-1} \sum_{m=1}^{n} Y_m(f)$$

and

$$\bar{\tau}(n) = n^{-1} \sum_{m=1}^{n} \tau_m,$$

where τ_m is the length of the mth cycle and $Y_m(f)$ is the integral of $f(X(\cdot))$ over the mth cycle. The quantity $z_{1-\gamma} = \Phi^{-1}(1 - \gamma)$, where Φ is the distribution function of a standardized normal random variable, $N(0, 1)$, and $s^2(n)$ is a strongly consistent point estimate for

$$\sigma^2(f) = \mathrm{var}\,(Y_1(f) - r(f)\tau_1).$$

Asymptotic confidence intervals are based on the central limit theorem (c.l.t.)

$$\frac{n^{1/2}\{\hat{r}(n) - r(f)\}}{\sigma(f)/E\{\tau_1\}} \Rightarrow N(0, 1)$$

as $n \to \infty$.

3. Steady state estimation for symmetric stochastic Petri nets

In this section we consider estimation of $r(f) = E\{f(X)\}$ for symmetric SPN models under the assumption that the function f is symmetric in the sense that

$$f(s) = f(\phi_S^l(s)) \tag{19}$$

for all $s \in S$ and all $l = 1, 2, \cdots, N$. [We set $\phi_S^1(s) = \phi_S(s)$ and

$$\phi_S^l(s) = \phi_S(\phi_S^{l-1}(s))$$

for all $s \in S$ and $l = 2, \cdots, N$. Similarly, we set $\phi_E^1(e) = \phi_E(e)$ and

$$\phi_E^l(e) = \phi_E(\phi_E^{l-1}(e))$$

for all $e \in E$.]

The key observation is that, under the assumptions of Proposition 4, regenerative cycles defined by the times at which transition $\phi_E^1(e^*)$ fires and the marking changes to $\phi_S^1(s_0')$ can be decomposed into independent, nonidentically distributed blocks. These blocks are determined by the successive times T_n at which transition $\phi_E^l(e^*)$ fires and the marking changes from $\phi_S^l(s^*)$ to $\phi_S^l(s_0')$ for some $s^* \in S^*$ and some l, $l = 1, 2, \cdots, N$. Estimates for $r(f)$ can be based on observation of these blocks. Proposition 5 provides conditions which ensure that (for each l) transition $\phi_E^l(e^*)$ fires and the marking changes to $\phi_S^l(s_0')$ infinitely often with probability one.

Let $\{T_n^l: n \geq 0\}$ be an increasing sequence of stopping times that are finite ($T_n^l < \infty$ a.s.) transition firing times such that for some $e^* \in E$ and $S^* \subseteq S$, $T_0^l = 0$ and

$$T_n^l = \inf\{t > T_{n-1}^l: \text{at time } t \text{ transition } \phi_E^l(e^*) \text{ fires}$$
$$\text{and the marking is } \phi_S^l(s^*) \text{ for some } s^* \in S^*\},$$

$n \geq 1$ and $l = 1, 2, \cdots, N$. Denote by $\{T_n: n \geq 1\}$ the times $T_1^1, T_1^2, \cdots, T_1^N, T_2^1, \cdots$ in increasing order.

Proposition 5
Suppose there exists $\delta > 0$ such that

$$P\{X(T_n^1) = \phi_S^1(s_0') \mid X(T_{n-1}^1), \cdots, X(T_1^1)\} \geq \delta \text{ a.s.} \tag{20}$$

Then $P\{X(T_n) = \phi_S^l(s_0') \text{ i.o.}\} = 1$ for all $l = 1, 2, \cdots, N$.

Proof By Lemma 4 of [8] it suffices to show that

$$P\{X(T_n^l) = \phi_S^l(s_0') \mid X(T_{n-1}^l), \cdots, X(T_1^l)\} \geq \delta \text{ a.s.} \tag{21}$$

Let e_n^* denote the transition that fires at time ζ_n, $n \geq 0$. The definition of a symmetric SPN implies that for all $x_0, x_1, \cdots, x_n \in S$ and $e_{i_1}, \cdots, e_{i_n} \in E$,

$$\begin{aligned} P\{X(\zeta_n) &= \phi_S^1(x_n), e_n^* = \phi_E^1(e_{i_n}), \cdots, X(\zeta_1) = \phi_S^1(x_1), \\ &\quad e_1^* = \phi_E^1(e_{i_1}), X(\zeta_0) = \phi_S^1(x_0)\} \\ &= P\{X(\zeta_n) = \phi_S^l(x_n), e_n^* = \phi_E^l(e_{i_n}), \cdots, X(\zeta_1) = \phi_S^l(x_1), \\ &\quad e_1^* = \phi_E^l(e_{i_1}), X(\zeta_0) = \phi_S^l(x_0)\}, \end{aligned} \tag{22}$$

$l = 1, 2, \cdots, N$. Denote by $\{\gamma_n^l: n \geq 0\}$ the indices of the successive times $\{\zeta_n: n \geq 0\}$ at which transition $\phi_E^l(e^*)$ fires when the marking is $\phi_S^l(s^*)$ for some $s^* \in S^*$. Equation (22) implies that for all $x_1, \cdots, x_n \in S$ and $s_1^*, \cdots, s_n^* \in S^*$,

$$\begin{aligned} P\{X(\zeta_{\gamma_n^1}) &= \phi_S^1(x_n), e_n^* = \phi_E^1(e^*), X(\zeta_{\gamma_n^1-1}) = \phi_S^1(s_n^*), \cdots, \\ &\quad X(\zeta_{\gamma_1^1}) = \phi_S^1(x_1), e_1^* = \phi_E^1(e^*), X(\zeta_{\gamma_1^1-1}) = \phi_S^1(s_1^*)\} \\ &= P\{X(\zeta_{\gamma_n^l}) = \phi_S^l(x_n), e_n^* = \phi_E^l(e^*), X(\zeta_{\gamma_n^l-1}) = \phi_S^l(s_n^*), \\ &\quad \cdots, X(\zeta_{\gamma_1^l}) = \phi_S^l(x_1), e_1^* = \phi_E^l(e^*), X(\zeta_{\gamma_1^l-1}) = \phi_S^l(s_1^*)\}, \end{aligned} \tag{23}$$

$l = 1, 2, \cdots, N$. Using the definition of $\{T_n^l: n \geq 0\}$, Equation (23) implies that for all $x_1, \cdots, x_n \in S$,

$$\begin{aligned} P\{X(T_n^1) &= \phi_S^1(x_n), \cdots, X(T_1^1) = \phi_S^1(x_1)\} \\ &= P\{X(T_n^l) = \phi_S^l(x_n), \cdots, X(T_1^l) = \phi_S^l(x_1)\}, \end{aligned} \tag{24}$$

$l = 1, 2, \cdots, N$. Applying Equation (24), it follows that for all $x_1, \cdots, x_{n-1} \in S$,

$$P\{X(T_n^1) = \phi_S^1(s_0') \mid X(T_{n-1}^1) = \phi_S^1(x_{n-1}), \cdots, X(T_1^1) = \phi_S^1(x_1)\}$$

$$= \frac{P\{X(T_n^1) = \phi_S^1(s_0'), X(T_{n-1}^1) = \phi_S^1(x_{n-1}), \cdots, X(T_1^1) = \phi_S^1(x_1)\}}{P\{X(T_{n-1}^1) = \phi_S^1(x_{n-1}) \cdots, X(T_1^1) = \phi_S^1(x_1)\}}$$

$$= \frac{P\{X(T_n^l) = \phi_S^l(s_0'), X(T_{n-1}^l) = \phi_S^l(x_{n-1}), \cdots, X(T_1^l) = \phi_S^l(x_1)\}}{P\{X(T_{n-1}^l) = \phi_S^l(x_{n-1}) \cdots, X(T_1^l) = \phi_S^l(x_1)\}}$$

$$= P\{X(T_n^l) = \phi_S^l(s_0') \mid X(T_{n-1}^l) = \phi_S^l(x_{n-1}), \cdots, X(T_1^l) = \phi_S^l(x_1)\},$$

$l = 1, 2, \cdots, N$. The result follows from Equation (20). □

To obtain estimates for $r(f)$, carry out the simulation of the marking process $\{X(t): t \geq 0\}$ in random blocks defined by the successive random times $\{T_{\beta_k}: k \geq 0\}$, where

$$T_{\beta_k} = \inf\{T_n^l > T_{\beta_{k-1}}: X(T_n^l) = \phi_S^l(s_0') \text{ for some } l, l = 1, 2, \cdots, N\}, \tag{25}$$

$k \geq 1$; $\beta_0 = 0$ and $T_{\beta_0} = 0$. [Note that the random times $\{T_{\beta_k}: k \geq 0\}$ do *not* form a sequence of regeneration points for the process $\{X(t): t \geq 0\}$.] Set $\tau_k = T_{\beta_k} - T_{\beta_{k-1}}$ and

$$Y_k(f) = \int_{T_{\beta_{k-1}}}^{T_{\beta_k}} f(X(s))ds,$$

for $k \geq 1$.

Proposition 6
The sequence $\{(Y_k(f), \tau_k): k \geq 1\}$ consists of independent and identically distributed pairs of random variables.

Proof The sequence $\{\beta_k: k \geq 1\}$ comprises indices of the successive stopping times $\{T_n: n \geq 1\}$ at which transition $\phi_E^l(e^*)$ fires and the marking changes from $\phi_S^l(s^*)$ to $\phi_S^l(s_0')$ for some $s^* \in S^*$ and some $l, l = 1, 2, \cdots, N$. Thus, by the definition of a symmetric SPN, each of the clocks running at time $T_{\beta_k}+$ was set or can be viewed as having been probabilistically reset at time T_{β_k}. Therefore $\{X(t): t \geq T_{\beta_k}\}$ determines the distribution of $\tau_{k+1} = T_{\beta_{k+1}} - T_{\beta_k}$ and the finite dimensional distributions of $\{f(X(t)): t \geq T_{\beta_k}\}$. Observe that the joint distribution of $X(T_{\beta_k}) = \phi_S^l(s_0')$, and the clocks set or reset at time T_{β_k} depend on the past history of the marking process only through the new marking $\phi_S^l(s_0')$, the previous marking $\phi_S^l(s_0)$, and the trigger transition $\phi_E^l(e^*)$; this implies that the cycle length τ_{k+1} and $\{f(X(t)): t \geq T_{\beta_k}\}$ are independent of $\{(Y_j(f), \tau_j): j \leq k\}$. It follows that the pairs of random variables $\{(Y_k(f), \tau_k): k \geq 1\}$ are mutually independent.

Next observe that

$$Y_k(f) = \sum_{m=\gamma_{k-1}}^{\gamma_k - 1} f(X(\zeta_m))[\zeta_{m+1} - \zeta_m], \tag{26}$$

where $\zeta_{\gamma_k} = T_{\beta_k}, k \geq 0$. Set $\delta_k = \gamma_k - \gamma_{k-1}, k \geq 1$. According to this definition, δ_k is the number of transitions that fire in the kth block. It is sufficient to show that for all $z_1, \cdots, z_n \geq 0, x_1, \cdots, x_n \in S$, and $n \geq 1$,

$$\begin{aligned} &P_1\{\delta_1 = n + 1, f(X(\zeta_n)) = f(x_n), \zeta_n \leq z_n, \\ &\qquad f(X(\zeta_{n-1})) = f(x_{n-1}), \zeta_{n-1} \leq z_{n-1}, \cdots, \zeta_1 \leq z_1\} \\ &= P_1\{\delta_{k+1} = n + 1, f(X(\zeta_{\gamma_k+n})) = f(x_n), \zeta_{\gamma_k+n} \leq z_n, \\ &\qquad f(X(\zeta_{\gamma_k+n-1})) = f(x_{n-1}), \zeta_{\gamma_k+n-1} \leq z_{\gamma_k+n-1}, \cdots, \zeta_{\gamma_k+1} \leq z_1\}, \end{aligned} \tag{27}$$

$k \geq 1$. Here $P_1\{\cdot\}$ denotes the conditional probability associated with starting the SPN with marking $\phi_S^1(s_0')$ and all active clocks reset at time $t = 0$ according to the distributions

$$P\{C_{0,i} \leq c\} = F(c; \phi_S^1(s_0'), e_i, \phi_S^1(s_0), \phi_E^1(e^*)) \tag{28}$$

for $c \geq 0$, $e_i \in E(\phi_S^1(s_0'))$. Recall that $C_{0,i}$ is the clock reading associated with transition e_i at time 0. Denoting by $P_l\{\cdot\}$ the corresponding conditional probability when the initial marking is $\phi_S^l(s_0')$, the definition of a symmetric SPN implies that, for all $z_1, \cdots, z_n \geq 0, x_1, \cdots, x_n \in S$, and $e_{i_1}, \cdots, e_{i_n} \in E$ with $p(s_k; s_{k-1}, e_{i_k}) > 0$,

$$\begin{aligned} &P_1\{X(\zeta_n) = \phi_S^1(x_n), \zeta_n \leq z_n, e_n^* = \phi_E^1(e_{i_n}), X(\zeta_{n-1}) = \phi_S^1(x_{n-1}), \\ &\qquad \zeta_{n-1} \leq z_{n-1}, e_{n-1}^* = \phi_E^1(e_{i_{n-1}}), \cdots, e_1^* = \phi_E^1(e_{i_1})\} \\ &= P_l\{X(\zeta_n) = \phi_S^l(x_n), \zeta_n \leq z_n, e_n^* = \phi_E^l(e_{i_n}), \\ &\qquad X(\zeta_{n-1}) = \phi_S^l(x_{n-1}), \zeta_{n-1} \leq z_{n-1}, \\ &\qquad e_{n-1}^* = \phi_E^l(e_{i_{n-1}}), \cdots, e_1^* = \phi_E^l(e_{i_1})\}, \end{aligned}$$

$l = 1, 2, \cdots, N$. Hence, by Equation (19),

$$\begin{aligned} &P_1\{\delta_1 = n + 1, f(X(\zeta_n)) = f(x_n), \zeta_n \leq z_n, \\ &\qquad f(X(\zeta_{n-1})) = f(x_{n-1}), \zeta_{n-1} \leq z_{n-1}, \cdots, \zeta_1 \leq z_1\} \\ &= P_l\{\delta_1 = n + 1, f(X(\zeta_n)) = f(x_n), \zeta_n \leq z_n, f(X(\zeta_{n-1})) \\ &\qquad = f(x_{n-1}), \zeta_{n-1} \leq z_{n-1}, \cdots, \zeta_1 \leq z_1\} \end{aligned} \tag{29}$$

for all $z_1, \cdots, z_n \geq 0, x_1, \cdots, x_n \in S$, and $n \geq 1$. But by the independence argument of the first part of the proof

$$\begin{aligned} &P_1\{\delta_{k+1} = n + 1, f(X(\zeta_{\gamma_k+n})) = f(x_n), \zeta_{\gamma_k+n} \leq z_n, \\ &\qquad f(X(\zeta_{\gamma_k+n-1})) = f(x_{n-1}), \zeta_{\gamma_k+n-1} \leq z_{n-1}, \cdots, \\ &\qquad \zeta_{\gamma_k+1} \leq z_1, X(\zeta_{\gamma_k}) = \phi_S^l(s_0')\} \\ &= P_l\{\delta_1 = n + 1, f(X(\zeta_n)) = f(x_n), \zeta_n \leq z_n, \\ &\qquad f(X(\zeta_{n-1})) = f(x_{n-1}), \zeta_{n-1} \leq z_{n-1}, \cdots, \\ &\qquad \zeta_1 \leq z_1\} P_1\{X(\zeta_{\gamma_k}) = \phi_S^l(s_0')\}, \end{aligned} \tag{30}$$

for all $z_1, \cdots, z_n \geq 0, x_1, \cdots, x_n \in S$, and $n \geq 1$. Applying Equation (30) and then Equation (29), it follows that, for all $z_1, \cdots, z_n \geq 0$ and $x_1, \cdots, x_n \in S$,

$$
\begin{aligned}
&P_1\{\delta_{k+1} = n+1, f(X(\zeta_{\gamma_k+n})) = f(x_n),\ \zeta_{\gamma_k+n} \le z_n,\\
&\quad f(X(\zeta_{\gamma_k+n-1})) = f(x_{n-1}),\ \zeta_{\gamma_k+n-1} \le z_{n-1}, \cdots, \zeta_{\gamma_k+1} \le z_1\}\\
&= \sum_{l=1}^{N} P_l\{\delta_1 = n+1, f(X(\zeta_n)) = f(x_n),\ \zeta_n \le z_n,\\
&\quad f(X(\zeta_{n-1})) = f(x_{n-1}),\ \zeta_{n-1} \le z_{n-1}, \cdots,\\
&\quad \zeta_1 \le z_1\} P_1\{X(\zeta_{\gamma_k}) = \phi_S^l(s_0')\}\\
&= \sum_{l=1}^{N} P_1\{\delta_1 = n+1, f(X(\zeta_n)) = f(x_n),\ \zeta_n \le z_n,\\
&\quad f(X(\zeta_{n-1})) = f(x_{n-1}),\ \zeta_{n-1} \le z_{n-1}, \cdots,\\
&\quad \zeta_1 \le z_1\} P_1\{X(\zeta_{\gamma_k}) = \phi_S^l(s_0')\}\\
&= P_1\{\delta_1 = n+1, f(X(\zeta_n)) = f(x_n),\ \zeta_n \le z_n,\\
&\quad f(X(\zeta_{n-1})) = f(x_{n-1}),\ \zeta_{n-1} \le z_{n-1}, \cdots, \zeta_1 \le z_1\},
\end{aligned}
$$

so that [using Equation (26)] the pairs of random variables $\{(Y_k(f), \tau_k): k \ge 1\}$ are identically distributed. □

Standard arguments establish a ratio formula for $r(f)$.

Proposition 7

Provided that $E\{\tau_1\} < \infty$ and $E\{|f(X)|\} < \infty$,

$$r(f) = \frac{E\{Y_1(f)\}}{E\{\tau_1\}}.$$

With these results Equations (17) and (18) provide point estimates and confidence intervals for $r(f)$. Propositions 6 and 7 assert that there are "independent, identically distributed pairs" and an appropriate "ratio formula" in the setting of independent, nonidentically distributed blocks. These results are sufficient to establish the validity of the confidence interval obtained in regenerative simulation theory.

Example 8

In the token ring model of Example 3, set $S' = \{s \in S: s_{j,3} = 1 \text{ for some } j, j = 1, 2, \cdots, N\}$ and consider the function f defined by

$$f(s) = 1_{\{S'\}}(s)$$

for $s \in S$. According to this definition, $r(f)$ is the steady state throughput of the token ring. Note that the function f satisfies Equation (19) since (for each l)

$$s_{\phi^l(j),3} = 1 \text{ if and only if } s_{j+l,3} = 1,$$

$j = 1, 2, \cdots, N$.

Set $e^* = e_{1,3}$, $s_0' = (1, 0, 1, 0, 1, 0, 0, 0, \cdots, 1, 0, 0, 0)$, and

$$S^* = \{(s_{1,1}, s_{1,2}, \cdots, s_{N,4}) \in S: s_{N,4} = 1\}$$

so that T_n^1 is the nth time that port 1 observes the ring token. Arguments given in [8] show that the conditions of Proposition 4 hold provided that the packet interarrival time random variables, A_j, have "new better than used" (NBU) distributions and satisfy a positivity condition: $P\{A_j \le R_N\} > 0, j = 1, 2, \cdots, N$. Since $\phi_E^l(e^*) = e_{l,3}$ and $\phi_S^l(s_0')$ is the marking in which there is a packet queued for transmission at each of the ports and the ring token is propagating to port l, it follows that T_n^l is the nth time that port l observes the ring token. We carry out the simulation of the marking process in blocks defined by the random times T_n at which for some l there is a packet queued for transmission at ports $1, 2, \cdots, l-1, l+1, \cdots, N$ and port l starts transmission of a packet.

4. Passage times in stochastic Petri nets

Formal specification of passage times in a symmetric SPN is by means of four subsets (A_1, A_2, B_1, and B_2) of the marking set, S. The sets A_1, A_2, B_1, and B_2 in effect determine when to start and stop the clock measuring a particular passage time; cf. Iglehart and Shedler [17].

Denoting the jump times of the process $\{X(t): t \ge 0\}$ by $\{\zeta_n: n \ge 0\}$, for $k, n \ge 1$, we require that the sets A_1, A_2, B_1, and B_2 satisfy the conditions

if $X(\zeta_{n-1}) \in A_1$, $X(\zeta_n) \in A_2$, $X(\zeta_{n-1+k}) \in A_1$, and

$X(\zeta_{n+k}) \in A_2$,

then $X(\zeta_{n-1+m}) \in B_1$ and $X(\zeta_{n+m}) \in B_2$

for some $0 < m \le k$;

and

if $X(\zeta_{n-1}) \in B_1$, $X(\zeta_n) \in B_2$, $X(\zeta_{n-1+k}) \in B_1$, and

$X(\zeta_{n+k}) \in B_2$,

then $X(\zeta_{n-1+m}) \in A_1$ and $X(\zeta_{n+m}) \in A_2$

for some $0 \le m < k$.

These conditions ensure that the start and termination times for the specified passage time strictly alternate.

In terms of the sets A_1, A_2, B_1, and B_2, define $A_1^l = \{\phi_S^l(s): s \in A_1\}$, $A_2^l = \{\phi_S^l(s): s \in A_2\}$, $B_1^l = \{\phi_S^l(s): s \in B_1\}$, and $B_2^l = \{\phi_S^l(s): s \in B_2\}$, $l = 1, 2, \cdots, N$. [Recall that for $s \in S$, $\phi_S^1(s) = \phi_S(s)$ and $\phi_S^l(s) = \phi(\phi_S^{l-1}(s))$, $l = 1, 2, \cdots, N$. Also recall that for $e \in E$, $\phi_E^1(e) = \phi_E(e)$ and $\phi_E^l(e) = \phi_E(\phi_E^{l-1}(e))$.] Then define two sequences of random times $\{S_j(l): j \ge 0\}$ and $\{T_j(l): j \ge 1\}$: $S_{j-1}(l)$ is the start time for the jth passage time (corresponding to the sets A_1^l, A_2^l, B_1^l, and B_2^l) and $T_j(l)$ is the termination time of this jth passage time. Set

$$S_0(l) = 0,$$

$$S_j(l) = \inf\{\zeta_n \ge T_j(l): X(\zeta_n) \in A_2^l, X(\zeta_{n-1}) \in A_1^l\},$$

and

$$T_j(l) = \inf\{\zeta_n > S_{j-1}(l): X(\zeta_n) \in B_2^l, X(\zeta_{n-1}) \in B_1^l\},$$

$j \geq 1$. Note that since the SPN is symmetric, the sets A_1^l, A_2^l, B_1^l, and B_2^l satisfy the conditions which ensure that the start and termination times for the passage times $P_j^l = T_j(l) - S_{j-1}(l)$ strictly alternate. Denote the successive passage times $P_1^1, P_1^2, \cdots, P_1^N, P_2^1, P_2^2, \cdots$ enumerated in termination order by $\{P_j': j \geq 1\}$. Set $T_0' = 0$ and let T_j' be the termination time for $P_j', j \geq 1$.

Proposition 9 gives conditions which ensure that $\{(X(T_n'), P_{n+1}'): n \geq 0\}$ is a regenerative process in discrete time and that the expected time between regeneration points is finite. The regenerative structure guarantees (Miller [18]) that $P_n' \Rightarrow P$ as $n \rightarrow \infty$. The goal of the simulation is the estimation of $r(f) = E\{f(P)\}$, where f is a real-valued (measurable) function and P is the limiting passage time.

We postulate the existence of a transition e^* and markings $s_0 \in B_1$ and $s_0' \in B_2$ such that a passage time terminates when transition e^* fires and the marking changes from s_0 to s_0'. In addition, we assume that no passage times are underway when the marking of the SPN is s_0'. Formally, let $L(t)$ be the last marking of the SPN before changing to $X(t)$ and set

$$V(t) = (L(t), X(t)).$$

Denote by G the state space of $\{V(t): t \geq 0\}$. Set $A^l = A_1^l \times A_2^l$ and $B^l = B_1^l \times B_2^l$, $l = 1, 2, \cdots, N$. Now set

$$H_1^l = \{s' \in S: (s, s') \in B^l - A^l \text{ for some } s \in S\},$$

so that H_1^l is the set of all possible markings when a passage time P_n^l terminates. Also set

$$H_2^l = \{s' \in S: (s, s') \in G - (B^l \cup A^l), (s_1', s_2') \overset{A^l}{\sim} (s, s'), \text{ and } (s, s') \overset{A^l}{\sim} (s_1'', s_2'') \text{ for all } s \in S \text{ and some } (s_1', s_2') \in B^l, (s_1'', s_2'') \in A^l\},$$

so that H_2^l is the set of all possible markings when a passage time P_n^l is not underway. For $(s, s'), (\bar{s}, \bar{s}') \in G$ we write $(s, s') \overset{A^l}{\sim} (\bar{s}, \bar{s}')$ if there exists a finite sequence $e_0', s_1', e_1', s_2', \cdots, s_n', e_n'$ of transitions and markings such that

$$p(s_1'; s', e_0')p(s_2'; s_1', e_1') \cdots p(\bar{s}; s_n', e_n') > 0$$

and $(s', s_1'), (s_n', \bar{s}), (s_j', s_{j+1}') \notin A^l, j = 1, 2, \cdots, n-1$. We assume that

$$B_2' = B_2 \cap (H_1^1 \cup H_2^1) \cap \cdots \cap (H_1^N \cup H_2^N) \neq \varnothing$$

and that $s_0' \in B_2'$.

As in Section 2, let $\{T_n^l: n \geq 0\}$ be an increasing sequence of stopping times that are finite ($T_n^l < \infty$ a.s.) transition firing times, such that for some $e^* \in E$ and $S^* \subseteq S$, $T_0^l = 0$ and

$$T_n^l = \inf\{t > T_{n-1}^l: \text{at time } t \text{ transition } \phi_E^l(e^*) \text{ fires and the marking is } \phi_S^l(s^*) \text{ for some } s^* \in S^*\},$$

$n \geq 1$ and $l = 1, 2, \cdots, N$.

Proposition 9

Suppose that there exist $e^* \in E$, $s_0 \in B_1$, and $s_0' \in B_2'$ such that $p(s_0'; s_0, e^*) > 0$ and either (i) $O(s_0'; s_0, e^*) = E(s_0') \cap (E(s_0) - \{e^*\}) = \varnothing$ or (ii) $O(s_0'; s_0, e^*) \neq \varnothing$ and for any $e' \in O(s_0'; s_0, e^*)$ the clock setting distribution $F(\cdot; s', e', s, e)$ is exponential, independent of s, s', and e. Set $v_0^1 = (\phi_S^1(s_0), \phi_S^1(s_0'))$ and suppose there exists $\delta > 0$ such that

$$P\{V(T_n^1) = v_0^1 \mid V(T_{n-1}^1), \cdots, V(T_1^1)\} \geq \delta \text{ a.s.} \tag{31}$$

Then $\{(X(T_n'), P_{n+1}'): n \geq 0\}$ is a regenerative process in discrete time. Moreover, if

$$E\{T_{n+1}^1 - T_n^1\} \leq c < \infty$$

for all $n \geq 1$, then the expected time between regeneration points is finite.

Proof Since $T_n^1 < \infty$ a.s. and $P\{V(T_n^1) = v_0^1 \mid V(T_{n-1}^1), \cdots, V(T_1^1)\} \geq \delta > 0$, Lemma 4 of [8] ensures that transition $\phi_E^1(e^*)$ fires and the marking of the SPN changes from $\phi_S^1(s_0)$ to $\phi_S^1(s_0')$ infinitely often with probability one: $P\{V(T_n^1) = v_0^1$ i.o.$\} = 1$. Denote by $\{\beta_k^1: k \geq 1\}$ the indices of the successive passage times $\{P_n': n \geq 1\}$ which terminate when transition $\phi_E^1(e^*)$ fires and the marking changes from $\phi_S^1(s_0)$ to $\phi_S^1(s_0')$. Let $T_0' = \beta_0^1 = 0$. We must show that

(i) $\{\beta_k^1: k \geq 0\}$ is a renewal process in discrete time,

and that for any $i_1 < i_2 < \cdots < i_m$ ($m \geq 1$) and $k \geq 0$,

(ii) $\{X(T'_{\beta_k^1+i_1}), P'_{\beta_k^1+i_1+1}, \cdots, X(T'_{\beta_k^1+i_m}), P'_{\beta_k^1+i_m+1}\}$ and $\{X(T'_{i_1}), P'_{i_1+1}, \cdots, X(T'_{i_m}), P'_{i_m+1}\}$ have the same distribution, and $\{X(T'_{\beta_k^1+i_1}), P'_{\beta_k^1+i_1+1}, \cdots, X(T'_{\beta_k^1+i_m}), P'_{\beta_k^1+i_m+1}\}$ is independent of $\{(X(T_n'), P_{n+1}'): 0 \leq n < \beta_k^1\}$.

At time $T'_{\beta_k^1}$, a passage time has just terminated with no other passage times underway. Now observe that each of the clocks running at time $T'_{\beta_k^1}+$ was set or can be viewed as having been probabilistically reset at time $T'_{\beta_k^1}$. [Assumption (ii) ensures that, no matter when the clock for transition $e' \in O(\phi_S^1(s_0'); \phi_S^1(s_0), \phi_E^1(e^*))$ was set, the remaining time until transition e' fires is exponentially distributed with the same parameter.] Therefore, $\{X(t): t \geq T'_{\beta_k}\}$ determines the finite dimensional distributions of $X(T'_{\beta_k^1+i}), P'_{\beta_k^1+i+1}$ for $i \geq 0$ and the distribution of $\beta_{k+1}^1 - \beta_k^1$. The joint distribution of $X(T'_{\beta_k^1})$ and the clocks set or reset at time $T'_{\beta_k^1}$ depends on the past history of the SPN only through $\phi_S^1(s_0')$, the previous marking $\phi_S^1(s_0)$, and the trigger transition $\phi_E^1(e^*)$. This distribution is the same for all β_k^1 and therefore (i) and (ii) hold.

Proposition 4.3 of [8] implies that $\{X(t): t \geq 0\}$ is a regenerative process in continuous time and that $E\{T'_{\beta_{k+1}^1} - T'_{\beta_k^1}\} < \infty$. It follows, since the state space of the marking process is finite and the clock setting distributions have finite mean, that $E\{\beta^1_{k+1} - \beta^1_k\} < \infty$. □

Proposition 10 provides sufficient conditions which ensure that Equation (31) holds. We postulate the existence of a distinguished random time T^+_{n-1} in the interval $[T^1_{n-1}, T^1_n)$ and a set $\{e^{(k)}: k \in K(v^+_n)\}$ of distinguished transitions determined by the marking, v^+_n, at time T^+_{n-1}. We make the following *sample path assumption*: $V(T^1_n) = v^1_0$ when each of the distinguished transitions occurs prior to some time $T^+_{n-1} + R_{n,k}(v^+_n)$. Proposition 10 asserts that the geometric trials recurrence criterion [Equation (34)] is satisfied if the clock setting distributions associated with the distinguished transitions are NBU and satisfy a "positivity" condition [condition (iii)] which guarantees the existence of $\delta > 0$ as in Equation (34). [A positive random variable A is NBU if

$$P\{A > x + y \mid A > y\} \leq P\{A > x\}$$

for all $x, y \geq 0$. Note that every increasing failure rate (IFR) distribution is NBU. Also, if A and B are independent random variables with NBU distributions, then the distributions of $A + B$, $\min(A, B)$, and $\max(A, B)$ are NBU.]

Recall that G is the state space of the process $\{V(t): t \geq 0\}$. Let $\{T^+_{n-1}: n \geq 0\}$ be a sequence of transition firing times and denote the state space of $\{V(T^+_{n-1}): n \geq 0\}$ by G^+. Set $\mathcal{H}(T^+_{n-1}) = \{(S_l, C_l): 0 \leq l < N(T^+_{n-1})\}$, where $N(\cdot)$ is given by Equation (7). Let $e^{(1)}, e^{(2)}, \cdots, e^{(m)} \in E$ and for $v^+ = (l^+, x^+) \in G^+$, set $E(v^+) = E(x^+)$ and

$$K(v^+) = \{k: e^{(k)} \in E(v^+)\}.$$

When $V(T^+_{n-1}) = v^+$, for $k \in K(v^+)$ we denote by $S_{n,k}(v^+)$ the latest time less than or equal to T^+_{n-1} at which the clock associated with transition $e^{(k)}$ was set and by $A_{n,k}(v^+)$ the setting on the clock at time $S_{n,k}(v^+)$.

Proposition 10
Let $e^{(1)}, e^{(2)}, \cdots, e^{(m)} \in E$ and let $\{T^+_{n-1}: n \geq 0\}$ be a sequence of transition firing times. For $v^+ \in G^+$, let $\{R_{n,k}(v^+): k \in K(v^+)\}$ be identically distributed collections of random variables, independent of $\{A_{n,k}(V(T^+_{n-1})): k \in K(V(T^+_{n-1}))\}$ and $\mathcal{H}(T^+_{n-1})$. Assume that

(i) $T^1_{n-1} \leq T^+_{n-1}$ a.s. and for $v_0, v_1, \cdots, v_{n-1} \in G$ and $v^+ \in G^+$,

$$\begin{aligned} &P\{V(T^1_n) = v^1_0,\ V(T^+_{n-1}) = v^+,\ V(T^1_{n-1}) = v_{n-1}, \cdots, \\ &\quad V(T^1_1) = v_0\} \\ &\geq P\{S_{n,k}(v^+) + A_{n,k}(v^+) \leq T^+_{n-1} + R_{n,k}(v^+),\ k \in K(v^+); \\ &\quad V(T^+_{n-1}) = v^+,\ V(T^1_{n-1}) = v_{n-1}, \cdots, V(T^1_1) = v_0\}; \end{aligned} \tag{32}$$

(ii) for all $e^{(k)}$ the clock setting distribution $F(\cdot; s', e^{(k)}, s, e) = F(\cdot; e^{(k)})$ and is NBU; and

(iii) there exists $\delta > 0$ such that for $v^+ \in G^+$

$$\delta(v^+) = P\{A_k(v^+) \leq R_{n,k}(v^+),\ k \in K(v^+)\} \geq \delta, \tag{33}$$

where the random variable $A_j(v^+)$ has distribution $F(\cdot; e^{(j)})$ and $\{A_j(v^+): j \in K(v^+)\}$ are mutually independent and independent of $\{R_{n,j}(v^+): j \in K(v^+)\}$.

Then

$$P\{V(T^1_n) = v^1_0 \mid V(T^1_{n-1}), \cdots, V(T^1_1)\} \geq \delta \text{ a.s.}, \tag{34}$$

so that $P\{V(T^1_n) = v^1_0 \text{ i.o.}\} = 1$.

Proposition 10 follows directly from Proposition 2.16 of [9] since the process $\{V(t): t \geq 0\}$ is a generalized semi-Markov process with state space, G, and event set, E.

Example 11
In the token ring model of Example 3, take $\phi(j) = j + 1$, $j = 1, 2, \cdots, N$. Set $s_0 = (1, 0, 0, 0, \cdots, 1, 0, 0, 0, 1, 0, 0, 1)$ and $s'_0 = (1, 0, 1, 0, 1, 0, 0, 0, \cdots, 1, 0, 0, 0)$. Take $e^* = e_{1,3}$ and

$$S^* = \{(s_{1,1}, s_{1,2}, \cdots, s_{N,4}) \in S: s_{N,4} = 1\}$$

[where S is given in Equation (11)] so that T^1_n is the nth time at which port 2 observes the ring token, $n \geq 1$. [Note that $X(T^1_n) = \phi^1_S(s'_0)$ if there is a packet queued for transmission at each of the other ports and port 2 starts transmission of a packet at time T^1_n. The SPN with marking process $\{X(t): t \geq 0\}$ changes marking to $\phi^1_S(s'_0)$ when transition $\phi^1_E(e^*)$ fires and the current marking is $\phi^1_S(s_0)$.] Observe that $T^1_n < \infty$ a.s. since

$$E\{T^1_n - T^1_{n-1}\} \leq NR_1 + NE\{L_1\} < \infty$$

for all $n \geq 1$. Take $e^{(j)} = e_{j,1} (m = N)$. Let T^+_{n-1} be the first time after T^1_{n-1} at which the ring token leaves port 1 [transition $\phi^1_E(e^*)$ becomes enabled]. Take $R_{n,k}(v^+) = R_1$ for all $v^+ \in G^+$. Since the SPN has marking $\phi^1_S(s'_0)$ at time T^1_n, if each transition $e_{j,1}$ enabled at time T^+_{n-1} fires before $\phi^1_E(e_3)$ fires, condition (i) of Proposition 10 is satisfied. Assume that for $j = 1, 2, \cdots, N$: (i) the distribution of A_j is NBU and (ii) $\delta_j = P\{A_j \leq R_1\} > 0$ so that

$$\delta(v^+) = \prod_{j \in K(v^+)} \delta_j \geq \prod_{j=1}^{N} \delta_j = \delta > 0.$$

Then $P\{V(T^1_n) = v^1_0 \text{ i.o.}\} = 1$.

The definition of a symmetric SPN implies for the process $\{(X(T'_n), P'_{n+1}): n \geq 0\}$ that regenerative cycles defined by the times at which the transition $\phi^1_E(e^*)$ fires and the marking changes from $\phi^1_S(s_0)$ to $\phi^1_S(s'_0)$ can be decomposed into independent, nonidentically distributed blocks. These blocks are defined by the successive times T_n at which transition $\phi^l_E(e^*)$ fires and the marking changes from $\phi^l_S(s^*)$ to $\phi^l_S(s'_0)$ for some $s^* \in S^*$ and some l, $l = 1, 2, \cdots, N$. Estimates for characteristics of limiting passage times can be based on measurement of passage times contained in these blocks. Denote the state space of the process $\{V(T^l_n): n \geq 0\}$ by G^l

and set

$$v_0^l = (\phi_S^l(s_0), \phi_S^l(s_0')), \tag{35}$$

$l = 1, 2, \cdots, N$. Denote by $\{T_n: n \geq 1\}$ the times $T_1^1, T_1^2, \cdots, T_1^N, T_2^1, \cdots$ in increasing order.

Proposition 12
Suppose there exists $\delta > 0$ such that

$$P\{V(T_n^1) = v_0^1 \mid V(T_{n-1}^1), \cdots, V(T_1^1)\} \geq \delta \text{ a.s.} \tag{36}$$

Then $P\{V(T_n) = v_0^l \text{ i.o.}\} = 1$ for all $l = 1, 2, \cdots, N$.

Arguments analogous to those given in Section 3 establish Proposition 12. Using symmetry of the SPN, the idea is to show that

$$\begin{aligned} &P\{V(T_n^1) = v_0^1 \mid V(T_{n-1}^1) = \phi_S^1(v_{n-1}), \cdots, V(T_1^1) = \phi_S^1(v_0)\} \\ &\quad = P\{V(T_n^l) = v_0^l \mid V(T_{n-1}^l) = \phi_S^l(v_{n-1}), \cdots, \\ &\qquad V(T_1^l) = \phi_S^l(v_0)\} \end{aligned} \tag{37}$$

for all $v_0, v_1, \cdots, v_{n-1} \in G^N$. [For $v = (s, s') \in G^N$, we write $\phi_S(v) = (\phi_S(s), \phi_S(s'))$.]

Carry out the simulation of $\{V(t): t \geq 0\}$ in random blocks defined by the successive random times $\{T'_{\beta_k}: k \geq 0\}$, where

$$\begin{aligned} T'_{\beta_k} = \inf\{&T_n^l > T'_{\beta_{k-1}}: V(T_n^l) = v_0^l \\ &\text{for some } l,\ l = 1, 2, \cdots, N\}, \end{aligned} \tag{38}$$

$k \geq 1$; $\beta_0 = 0$ and $T'_{\beta_0} = 0$. Each epoch T'_{β_k} corresponds to the termination of a passage time with no other passage times underway. [Note that the random times $\{T'_{\beta_k}: k \geq 0\}$ do *not* form a sequence of regeneration points for the process $\{(X(T'_n), P'_{n+1}): n \geq 0\}$.]

Set $\alpha_k = \beta_k - \beta_{k-1}$, $k \geq 1$. According to this definition α_k is the number of passage times in the kth block. Also set

$$Y_1(f) = \sum_{j=1}^{\alpha_1} f(P'_j)$$

and denote the analogous quantity in the kth block by $Y_k(f)$, $k \geq 1$.

Proposition 13
The sequence of pairs of random variables $\{(Y_k(f), \alpha_k): k \geq 1\}$ are independent and identically distributed.

Proof As in the proof of Proposition 9, observe that at time T'_{β_k} defined by Equation (38) a passage time has just terminated with no passage times underway and each of the clocks running at time $T'_{\beta_k}+$ was set or can be viewed as having been probabilistically reset at time T'_{β_k}. Therefore $\{X(t): t \geq T'_{\beta_k}\}$ determines the distribution of $\alpha_{k+1} = \beta_{k+1} - \beta_k$ and the finite dimensional distributions of P'_{β_k+i+1} for $i \geq 0$. The joint distribution of the clocks set or reset at time T'_{β_k} depends on the past history of the marking process only through $X(T'_{\beta_k}) = \phi_S^l(s_0')$, the previous marking $\phi_S^l(s_0)$, and the trigger transition $\phi_E^l(e^*)$. It follows that the pairs of random variables $\{(Y_k(f), \alpha_k): k \geq 1\}$ are mutually independent.

Recall that ζ_n is the time of the nth transition firing and denote by e_n^* the transition that fires at time ζ_n, $n \geq 0$. Also recall that C_n is the vector of clock readings at time ζ_n and that $C_{n,i}$ is the ith coordinate of the vector C_n for $e_i \in E(S_n)$. Let $z_1, \cdots, z_n \geq 0$, $x_1, \cdots, x_n \in S$, and $e_{i_1}, \cdots, e_{i_n} \in E$ with $p(x_k; x_{k-1}, e_{i_k}) > 0$. It follows from the definition of a symmetric SPN that

$$\begin{aligned} P_1\{&X(\zeta_n) = \phi_S^1(x_n),\ \zeta_n \leq z_n,\ e_n^* = \phi_E^1(e_{i_n}),\ X(\zeta_{n-1}) = \phi_S^1(x_{n-1}), \\ &\zeta_{n-1} \leq z_{n-1},\ e_{n-1}^* = \phi_E^1(e_{i_{n-1}}), \cdots, e_1^* = \phi_E^1(e_{i_1})\} \\ = P_l\{&X(\zeta_n) = \phi_S^l(x_n),\ \zeta_n \leq z_n,\ e_n^* = \phi_E^l(e_{i_n}), \\ &X(\zeta_{n-1}) = \phi_S^l(x_{n-1}),\ \zeta_{n-1} \leq z_{n-1}, \\ &e_{n-1}^* = \phi_E^l(e_{i_{n-1}}), \cdots, e_1^* = \phi_E^l(e_{i_1})\} \end{aligned} \tag{39}$$

for all $l = 1, 2, \cdots, N$. [Here $P_1\{\cdot\}$ denotes the conditional probability associated with starting the SPN with marking $\phi_S^1(s_0')$ and all active clocks reset at time $t = 0$ according to the distributions

$$P\{C_{0,i} \leq c\} = F(c;\ \phi_S^1(s_0'),\ e_i,\ \phi_S^1(s_0),\ \phi_E^1(e^*))$$

for $c \geq 0$, $e_i \in E(\phi_S^1(s_0'))$; $P_l\{\cdot\}$ denotes the corresponding conditional probability when the initial marking is $\phi_S^l(s_0')$.]

Next suppose that $X(0) = s_0'$ and that all active clocks are reset at time $t = 0$ according to the distributions $F(c; s_0', e_i, s_0, e^*)$, $e_i \in E(s_0')$. Set $X^1(t) = \phi_S^1(X(t))$ and $X^l(t) = \phi_S^l(X(t))$, $t \geq 0$. Observe that for each sample path of $\{X(t): t \geq 0\}$ and all $n \geq 0$,

$$X^1(\zeta_{n-1}) \in A_1^{m_1} = \{\phi_S^{m_1}(s): s \in A_1\} \text{ and}$$

$$X^1(\zeta_n) \in A_2^{m_1} = \{\phi_S^{m_1}(s): s \in A_2\}$$

for some m_1 if and only if $X^l(\zeta_{n-1}) \in A_1^{m_l}$ and $X^l(\zeta_n) \in A_2^{m_l}$ for some m_l. Similarly,

$$X^1(\zeta_{n-1}) \in B_1^{m_1} = \{\phi_S^{m_1}(s): s \in B_1\} \text{ and}$$

$$X^1(\zeta_n) \in B_2^{m_1} = \{\phi_S^{m_1}(s): s \in B_2\}$$

for some m_1 if and only if $X^l(\zeta_{n-1}) \in B_1^{m_l}$ and $X^l(\zeta_n) \in B_2^{m_l}$ for some m_l. Since

$$S_j(m) = \inf\{\zeta_n \geq T_j(m): X(\zeta_n) \in A_2^m,\ X(\zeta_{n-1}) \in A_1^m\}$$

and

$$T_j(m) = \inf\{\zeta_n > S_{j-1}(m): X(\zeta_n) \in B_2^m,\ X(\zeta_{n-1}) \in B_1^m\}$$

for all m, Equation (39) implies that

$$\begin{aligned} &P_1\{\alpha_1 = n + 1,\ P'_{n+1} \leq y_{n+1},\ P'_n \leq y_n, \cdots, P'_1 \leq y_1\} \\ &\quad = P_l\{\alpha_1 = n + 1,\ P'_{n+1} \leq y_{n+1},\ P'_n \leq y_n, \cdots, P'_1 \leq y_1\} \end{aligned} \tag{40}$$

for all $l = 1, 2, \cdots, N$, $y_1, y_2, \cdots, y_{n+1} \geq 0$, and $n \geq 0$. By the independence argument in the first part of the proof, it follows that

$$P_1\{\alpha_{k+1} = n+1, P'_{\beta_{k+1}} \le y_{n+1}, P'_{\beta_{k+1}-1} \le y_n, \cdots, P'_{\beta_k+1} \le y_1, X(T'_{\beta_k}) = \phi^l_S(s'_0)\}$$
$$= P_l\{\alpha_1 = n+1, P'_{n+1} \le y_{n+1}, P'_n \le y_n, \cdots, P'_1 \le y_1\} \times P_1\{X(T'_{\beta_k}) = \phi^l_S(s'_0)\} \quad (41)$$

for all $n \ge 0$ and $l = 1, 2, \cdots, N$. Using Equation (40), this implies

$$\sum_{l=1}^{N} P_1\{\alpha_{k+1} = n+1, P'_{\beta_{k+1}} \le y_{n+1}, P'_{\beta_{k+1}-1} \le y_n, \cdots, P'_{\beta_k+1} \le y_1, X(T'_{\beta_k}) = \phi^l_S(s'_0)\}$$
$$= \sum_{l=1}^{N} P_l\{\alpha_1 = n+1, P'_{n+1} \le y_{n+1}, P'_n \le y_n, \cdots, P'_1 \le y_1\} P_1\{X(T'_{\beta_k}) = \phi^l_S(s'_0)\}$$
$$= \sum_{l=1}^{N} P_1\{\alpha_1 = n+1, P'_{n+1} \le y_{n+1}, P'_n \le y_n, \cdots, P'_1 \le y_1\} P_1\{X(T'_{\beta_k}) = \phi^l_S(s'_0)\},$$

so that

$$P_1\{\alpha_{k+1} = n+1, P'_{\beta_{k+1}} \le y_{n+1}, P'_{\beta_{k+1}-1} \le y_n, \cdots, P'_{\beta_k+1} \le y_1\}$$
$$= P_1\{\alpha_1 = n+1, P'_{n+1} \le y_{n+1}, P'_n \le y_n, \cdots, P'_1 \le y_1\},$$

and the pairs of random variables $\{(Y_k(f), \alpha_k): k \ge 1\}$ are identically distributed. □

Standard arguments establish a ratio formula for $r(f) = E\{f(P)\}$.

Proposition 14
Provided that $E\{\tau_1\} < \infty$, $P\{P \in D(f)\} = 0$ and $E\{|f(P)|\} < \infty$,

$$E\{f(P)\} = \frac{E\{Y_1(f)\}}{E\{\alpha_1\}}.$$

With these results, based on n blocks (cf. Crane and Iglehart [16]) a strongly consistent point estimate for $r(f)$ is

$$\hat{r}(n) = \frac{\bar{Y}(n)}{\bar{\alpha}(n)}, \quad (42)$$

and an asymptotic $100(1-2\gamma)\%$ confidence interval is

$$\left[\hat{r}(n) - \frac{z_{1-\gamma}s(n)}{\bar{\alpha}(n)n^{1/2}}, \hat{r}(n) + \frac{z_{1-\gamma}s(n)}{\bar{\alpha}(n)n^{1/2}}\right], \quad (43)$$

where $s^2(n)$ is a strongly consistent point estimate for $\sigma^2(f) = \text{var}(Y_1(f) - r(f)\alpha_1)$. Asymptotic confidence intervals are based on the c.l.t.

$$\frac{n^{1/2}\{\hat{r}(n) - r(f)\}}{\sigma(f)/E\{\alpha_1\}} \Rightarrow N(0, 1) \quad (44)$$

as $n \to \infty$.

Example 15
In Example 3, consider port access times measured from the arrival of a packet for transmission by some port until the start of transmission by the port. This sequence of passage times is specified by the four subsets of S:

$A_1 = \{(s_{1,1}, \cdots, s_{N,4}) \in S: s_{1,1} = s_{1,3} = 0\}$,
$A_2 = \{(s_{1,1}, \cdots, s_{N,4}) \in S: s_{1,1} = 1 \text{ and } s_{1,3} = 0\}$,
$B_1 = \{(s_{1,1}, \cdots, s_{N,4}) \in S: s_{1,3} = 0 \text{ and } s_{N,4} = 1\}$,
and
$B_2 = \{(s_{1,1}, \cdots, s_{N,4}) \in S: s_{1,3} = 1 \text{ and } s_{N,4} = 0\}$.

The set of all possible markings when a passage time P^l_j terminates or is not underway is $H^l = \{(s_{1,1}, s_{1,2}, \cdots, s_{N,4}) \in S: s_{l,1}s_{l,3} = 1 \text{ or } s_{l,2} = 1\}$. Then $B'_2 \ne \emptyset$ and $s'_0 = (1, 0, 1, 0, 0, 1, 0, 0, \cdots, 0, 1, 0, 0) \in B'_2$. The random times $\{T'_{\beta_k}: k \ge 0\}$ correspond to terminations of port access times which occur when there is no packet queued for transmission at any of the ports. Propositions 13 and 14 hold provided that the packet interarrival time random variables are exponentially distributed. [The random time T^l_n is the nth time at which port $l+1$ observes the ring token, $n \ge 0$. Note that $\{V(T^l_n): n \ge 0\}$ is an irreducible, finite state discrete time Markov chain so that $P\{V(T^l_n) = v^l_0 \text{ i.o.}\} = 1$. It follows that $P\{V(T_n) = v^l_0 \text{ i.o.}\} = 1$ for all $l = 1, 2, \cdots, N$.]

5. Statistical efficiency

Section 4 provides two estimation procedures for passage times in a symmetric SPN. Each procedure rests on the assumption that there exist $e^* \in E$, $s_0 \in B_1$, and $s'_0 \in B'_2$ satisfying the conditions of Proposition 9. The regenerative structure guarantees that $P'_n \Rightarrow P$ as $n \to \infty$ and the goal of the simulation is the estimation of $r(f) = E\{f(P)\}$, where f is a real-valued measurable function. [We assume that the function f is such that $E\{|f(P)|\} < \infty$ and $P\{P \in D(f)\} = 0$ so that ratio formulas for $r(f)$ hold.]

Estimates for $r(f)$ can be based on measurement of passage times $\{P^1_n: n \ge 1\}$ and simulation of the underlying marking process of the SPN in regenerative cycles defined by the times T'_n at which $V(T'_n) = v^1_0$. Alternatively, exploiting properties of a symmetric SPN, estimates can be based on measurement of passage times $\{P'_n: n \ge 1\}$ and simulation of the underlying marking process in independent, nonidentically distributed blocks defined by the times T'_n at which $V(T'_n) \in \{v^1_0, \cdots, v^N_0\}$. This estimation procedure extracts more passage time information from a simulation of fixed length and should provide estimates for $r(f)$ that are relatively more accurate. In this section we verify that this is indeed the case by showing that the resulting confidence intervals are shorter.

For $t \ge 0$ let $m^1(t)$ be the number of passage times $\{P^1_n: n \ge 1\}$ completed in $(0, t]$ and denote by $\{\beta^1_k: k \ge 1\}$ the indices of the successive termination times $\{T'_n: n \ge 1\}$ at which $V(T'_n) = v_0^1$. Set

$$\alpha_k^1 = m^1(T'_{\beta_k^1}) - m^1(T'_{\beta_{k-1}^1}),$$

$$Y_k^1(f) = \sum_{j=m^1(T'_{\beta_{k-1}^1})+1}^{m^1(T'_{\beta_k^1})} f(P_j^1),$$

$k \ge 1$. Also set

$$(\sigma^1(f))^2 = \mathrm{var}\,(Y_1^1(f) - r(f)\alpha_1^1).$$

Then by Lemma 4.1 of Iglehart and Shedler [17],

$$\frac{t^{1/2}\left(\dfrac{1}{m^1(t)}\sum_{j=1}^{m^1(t)} f(P_j^1) - r(f)\right)}{(E\{\tau_1^1\})^{1/2}\sigma^1(f)/E\{\alpha_1^1\}} \Rightarrow N(0, 1) \tag{45}$$

as $t \to \infty$ provided that $E\{(\alpha_1^1)^2\} < \infty$ and $E\{(Y_1^1(|f|))^2\} < \infty$. Here $\tau_k^1 = T'_{\beta_k^1} - T'_{\beta_{k-1}^1}$. Since the numerator in this c.l.t. and the limit $[N(0, 1)]$ is independent of the transition $\phi_E^1(e^*)$ and the markings $\phi_S^1(s_0)$ and $\phi_S^1(s'_0)$ which define the cycles, so is the denominator; this is a consequence of the convergence of types theorem (Billingsley [19], Theorem 14.2). Thus, the quantity

$$e^1(f) = (E\{\tau_1^1\})^{1/2}\sigma^1(f)/E\{\alpha_1^1\}$$

is an appropriate measure of the statistical efficiency of the estimation procedure based on cycles.

Now let $m(t)$ be the number of passage times $\{P'_n: n \ge 1\}$ completed in $(0, t]$. Set

$$\alpha_k = m(T'_{\beta_k^1}) - m(T'_{\beta_{k-1}^1}),$$

$$Y_k(f) = \sum_{j=m(T'_{\beta_{k-1}^1})+1}^{m(T'_{\beta_k^1})} f(P'_j),$$

and

$$(\sigma(f))^2 = \mathrm{var}\,(Y_1(f) - r(f)\alpha_1).$$

Again using Lemma 4.1 of [17],

$$\frac{t^{1/2}\left(\dfrac{1}{m(t)}\sum_{j=1}^{m(t)} f(P'_j) - r(f)\right)}{(E\{\tau_1^1\})^{1/2}\sigma(f)/E\{\alpha_1\}} \Rightarrow N(0, 1) \tag{46}$$

as $t \to \infty$ provided that $E\{(\alpha_1)^2\} < \infty$ and $E\{(Y_1(|f|))^2\} < \infty$. Now observe that the numerator and the limit in this c.l.t. are independent of whether the passage times $\{P'_n: n \ge 1\}$ are measured in regenerative cycles [defined by transition $\phi_E^1(e^*)$ and markings $\phi_S^1(s_0)$ and $\phi_S^1(s'_0)$] or in blocks defined by $\phi_E^l(e^*)$, $\phi_S^l(s_0)$, and $\phi_S^l(s'_0)$ for all $l = 1, 2, \cdots, N$. Therefore,

$$e(f) = (E\{\tau_1^1\})^{1/2}\sigma(f)/E\{\alpha_1\}$$

is an appropriate measure of statistical efficiency of the estimation procedure based on blocks.

Note that when the passage times $\{P_n^1: n \ge 1\}$ are used to construct point and interval estimates for $r(f)$, the half-length of the confidence interval is proportional to $e^1(f)$, and when the passage times $\{P'_n: n \ge 1\}$ are used (with the same constant of proportionality), the half-length of the confidence interval is proportional to $e(f)$. Proposition 16 asserts that under mild regularity conditions on the function f, $e(f) \le e^1(f)$.

Proposition 16
For all functions f such that $E\{|f(P)|\} < \infty$ and $P\{P \in D(f)\} = 0$, $e(f) \le e^1(f)$.

Proof It is sufficient to show that

$$(\sigma(f))^2 \le N^2(\sigma^1(f))^2 \tag{47}$$

and

$$E\{\alpha_1\} = NE\{\alpha_1^1\}. \tag{48}$$

To establish Equation (47), for $t \ge 0$ set

$$W(t) = \sum_{j=1}^{m(t)} f(P'_j) - r(f)m(t).$$

Now observe that $\{X(t): t \ge 0\}$ is a regenerative process by Proposition 4 and that, with respect to this process, $\{W(t): t \ge 0\}$ is a cumulative process in the sense of Smith [20] with

$$E\{W(T'_{\beta_k^1}) - W(T'_{\beta_{k-1}^1})\} = E\{Y_k(f) - r(f)\alpha_k\} = 0.$$

Thus, by Theorem 8 of [20],

$$\lim_{t\to\infty} \frac{\mathrm{var}\,(W(t))}{t} = \frac{(\sigma(f))^2}{E\{\tau_1^1\}}. \tag{49}$$

Next recall that $\{P'_n: n \ge 1\}$ is the sequence of passage times $P_1^1, P_1^2, \cdots, P_1^N, P_2^1, P_2^2, \cdots$ enumerated in termination order and therefore

$$\sum_{j=1}^{m(t)} f(P'_j) - r(f)m(t) = \sum_{l=1}^{N}\left\{\sum_{j=1}^{m^l(t)} f(P_j^l) - r(f)m^l(t)\right\},$$

where $m^l(t)$ is the number of passage times $\{P_n^l: n \ge 1\}$ completed in $(0, t]$. Now set

$$W^l(t) = \sum_{j=1}^{m^l(t)} f(P_j^l) - r(f)m^l(t)$$

so that

$$W(t) = \sum_{l=1}^{N} W^l(t)$$

and by the Cauchy-Schwarz inequality

$$\mathrm{var}\,(W(t)) \le \sum_{l=1}^{N} \mathrm{var}(W^l(t)) + \sum_{j\ne l} \{\mathrm{var}(W^j(t))\mathrm{var}(W^l(t))\}^{1/2}$$

$$= \left[\sum_{l=1}^{N} \{\mathrm{var}(W^l(t))\}^{1/2}\right]^2.$$

Equation (47) follows, since

$$\lim_{t\to\infty} \frac{\mathrm{var}\,(W^l(t))}{t} = \frac{(\sigma^1(f))^2}{E\{\tau_1^1\}}$$

for $l = 1, 2, \cdots, N$. To see this, fix l and let $\{\beta_k^l: k \geq 1\}$ be the indices of the successive termination times $\{T_n': n \geq 1\}$ at which $V(T_n') = v_0^l$. Observe that $\{W^l(t): t \geq 0\}$ is a cumulative process, so that by Theorem 8 of [20],

$$\lim_{t\to\infty} \frac{\mathrm{var}\,(W^l(t))}{t} = \frac{(\sigma^1(f))^2}{E\{\tau_2^l\}},$$

where $\tau_k^l = T'_{\beta_k^l} - T'_{\beta_{k-1}^l}$ and

$$(\sigma^l(f))^2 = \mathrm{var}\,(Y_2^l(f) - r(f)\alpha_2^l),$$

with $\alpha_k^l = m^l(T'_{\beta_k^l}) - m^l(T'_{\beta_{k-1}^l})$ and

$$Y_k^l(f) = \sum_{j=m^l(T'_{\beta_{k-1}^l})+1}^{m^l(T'_{\beta_k^l})} f(P_j^l).$$

The definition of a symmetric SPN implies that $E\{\tau_2^l\} = E\{\tau_2^1\}$ and $(\sigma^l(f))^2 = \mathrm{var}(Y_2^1(f) - r(f)\alpha_2^1)$.

To establish Equation (48) set

$$m(t) = \sum_{l=1}^{N} m^l(t)$$

and observe that $\{m^l(t): t \geq 0\}$ and $\{m(t): t \geq 0\}$ are cumulative processes with respect to $\{X(t): t \geq 0\}$. Moreover,

$$\lim_{t\to\infty} \frac{E\{m(t)\}}{t} = \frac{E\{\alpha_1^1\}}{E\{\tau_1^1\}}$$

and

$$\lim_{t\to\infty} \frac{E\{m^l(t)\}}{t} = \frac{E\{\alpha_2^l\}}{E\{\tau_2^l\}}.$$

Again, since the SPN is symmetric, $E\{\tau_1^l\} = E\{\tau_2^1\}$ and $E\{\alpha_2^l\} = E\{\alpha_2^1\}$ so that

$$\lim_{t\to\infty} \frac{E\{m^l(t)\}}{t} = \frac{E\{\alpha_1^1\}}{E\{\tau_1^1\}},$$

$l = 1, 2, \cdots, N.$ □

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