UNIVERSITY OF ILLINOIS

DIGITAL COMPUTER

AUXILIARY
LIBRARY ROUTINE E 9 - 308

TITLE: Closed (Newton-Codes) and Open (Steffensen) Quadrature (Numerical Integration for tabulated function values)

TYPE: Closed subroutine

NUMBER OF WORDS: 15 + n (Closed Quadrature)
13 + n (Open Quadrature)
(n defined below)

TEMPORARY STORAGE: 0

DURATION: 2.9 + 1.4n milliseconds (Closed Quadrature)
2.9 + 1.4(n-2) milliseconds (Open Quadrature)

ACCURACY: $\pm 2^{-39}$ + truncation error (see below)

DESCRIPTION: This routine calculates an approximation to the integral, using tabulated function values,

$$\frac{1}{b-a} \int_a^b f(x)\, dx \approx \frac{1}{D} \sum_{k=0}^{n} N_k f_k,$$

where

$$f_k = f(a + k \frac{(b-a)}{n}).$$

The closed and open quadrature cases are distinguished by the fact that N_0 and N_n are zero for open quadrature, and non-zero for closed quadrature. This means that an approximation to the integral can be computed when the terminal function values are not known. The number n is the number of subintervals into which the original interval (b-a) is divided; thus there are n + 1 function values in the closed case, and n - 1 function values in the open case. The numbers D and N_k are chosen to give optimum results for equally spaced values of x within the region of integration.

USE: To use this routine, the programmer will determine the number of intervals in his region of integration, and the type of quadrature desired. The coefficients appropriate to his choice are then copied from the second part of the library tape immediately following the program. The routine is entered with the orders

```
p   | -- tF
    | 50 pF
----+------
p+1 | 26 xF
```

where x is the location of the first word of this routine. The program parameter t is the location of the first of the tabulated function values f_k; in the closed case, the function value f_k will be in location t + k. In the open case, f_1 is to be at t, f_2 at t + 1, ..., and f_{n-1} at t + n-2. (Because $N_0 = N_n = 0$ in the open case, the function values f_0 and f_n are neither needed nor used by the routine).

When control is returned to the right-hand order in p + 1, the computed integral will be in both the accumulator and quotient registers.

TRUNCATION ERROR: The maximum error introduced in the integration by using only a finite number of function values is given approximately by the following table:

n=number of intervals	Closed Case	Open Case
1	$-\frac{1}{12} f^{(2)}(z)$	———
2	$-3.5\times10^{-4} f^{(4)}(z)$	———
3	$-1.6\times10^{-4} f^{(4)}(z)$	$\frac{1}{36} f^{(2)}(z)$
4	$-5.2\times10^{-7} f^{(6)}(z)$	$3.1\times10^{-4} f^{(4)}(z)$
5	$-3.0\times10^{-7} f^{(6)}(z)$	$2.2\times10^{-4} f^{(4)}(z)$
6	$-6.4\times10^{-10} f^{(8)}(z)$	$1.1\times10^{-6} f^{(6)}(z)$
7	$-4.0\times10^{-10} f^{(8)}(z)$	$7.4\times10^{-7} f^{(6)}(z)$
8	$-5.9\times10^{-13} f^{(10)}(z)$	$2.1\times10^{-9} f^{(8)}(z)$
9	$-3.8\times10^{-13} f^{(10)}(z)$	$1.5\times10^{-9} f^{(8)}(z)$
10	$-4.2\times10^{-16} f^{(12)}(z)$	$2.7\times10^{-12} f^{(10)}(z)$
11	$-2.7\times10^{-16} f^{(12)}(z)$	$2.0\times10^{-12} f^{(10)}(z)$
12	$-2.2\times10^{-19} f^{(14)}(z)$	$2.5\times10^{-15} f^{(12)}(z)$

Here, $f^{(p)}(z)$ is the pth derivative of the function being integrated, evaluated at some point z in the interval (a,b).

NOTE 1: This routine is independent of a linear change of variable, so that the limits of integration a and b need not be specified.

NOTE 2: If the programmer does not need to use function values of equally spaced points (for example, the function values are evaluated by an auxiliary subroutine), then greater accuracy may be obtained by using Gaussian Quadrature (see library routines pertaining to Gaussian Quadrature).

REFERENCES:

B. P. Moors, "Valeur Approximative d'une Intégrale Défini"

J. F. Steffensen, "Interpolation"

DATE July 7, 1960

PROGRAMMED BY John Ehrman

APPROVED BY [signature]

nj

LOCATION	ORDER	NOTES PAGE 1 E 9
	OOK	
0	K5 F	
	42 11L	Plant link
1	46 5L	
	41 F	Plant function value address, clear sum box
2	L5 3L	
	42 4L	Set coefficient address
3	23 4L	
	00 14L	enter loop
4	S5 F	
	50 F	
5	74 F	
	L4 F	
6	40 F	
	L5 5L	Step function value list address
7	L4 6L	
	46 5L	
8	F5 4L	
	42 4L	Step coefficient address
9	L0 12L	
	36 4L	test for end
10	L5 F	
	66 13L	
11	S5 F	$\frac{1}{D} \sum f_k N_k$
	22 F	
12	35 F	
	50 L	End constant
13	00 F	
	00 F	D
14	00 F	
	00 F	Coefficients N_0 (closed), N_1 (open)
.		. . . to be
.		. . . copied
.		. . . from
		library
	00 F	tape.
	00 F	. N_n N_{n-2}