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SUBJECT: SMOOTHING OF SAMPLES OF A FUNCTION AND ITS DERIVATIVE IN THE PRESENCE OF NOISE

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Abstract: An analysis of errors arising in linear filtering and prediction of samples of a function and its derivative is performed by the method of inverse probability. The results, showing the dependence of the errors on data accuracy, are presented in a graphical form.

The following analysis is concerned with the problem of smoothing and prediction of a random time series when past and present sampled measurements of both the function and its first derivative are simultaneously available. The discussion is limited to functions generated by a specific statistical process requiring that the second derivative of the function remain constant over each sampling interval with a Gaussian probability density distribution about a zero mean (cf. Fig. 1). It is also assumed that the values of the second derivative in each sampling interval are independent of each other.

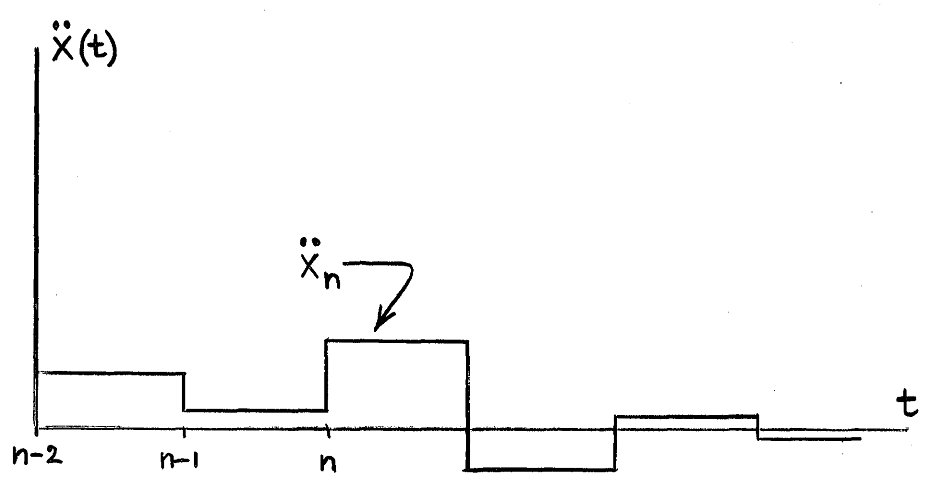

Figure 1

The probability density distribution function of the second derivative is

$$W(\ddot{x}_n) = \frac{1}{\sqrt{2\pi}\,\alpha} e^{-\frac{\ddot{x}_n^2}{2\alpha^2}} \tag{1}$$

where the standard deviation α is assumed to be known.

Since the second derivative is constant over each sampling interval, the sampled values of the first derivative and the function are

$$\dot{x}_n = \dot{x}_{n-1} + \ddot{x}_{n-1} \tag{2}$$

and

$$x_n = x_{n-1} + \dot{x}_{n-1} + \frac{1}{2}\ddot{x}_{n-1} \tag{3}$$

The noise in the samples of the function and its derivative is assumed to have independent Gaussian distributions. These are

$$W(N_n) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{N_n^2}{2\sigma^2}} \tag{4}$$

and

$$W(M_n) = \frac{1}{\sqrt{2\pi}\,\mu} e^{-\frac{M_n^2}{2\mu^2}} \tag{5}$$

in the function and the derivative respectively.

Fig. 2 shows a signal flow diagram of this problem in the complex frequency domain. Since only the second derivative is statistically prescribed, it is considered as the input in the diagram, while Eqs. (2) and (3) generating the samples of the derivative and the function are represented by appropriate linear operators. Here $z = e^{-s}$ denotes a unit delay operator. The corrupted samples f and d are fed into a smoothing filter F. Since both the signal and the noise have Gaussian distributions, the optimum smoothing filter must be linear and can be synthesized by the Wiener-Lee method. This analysis, however, illustrates a more direct approach by the use of inverse probability.

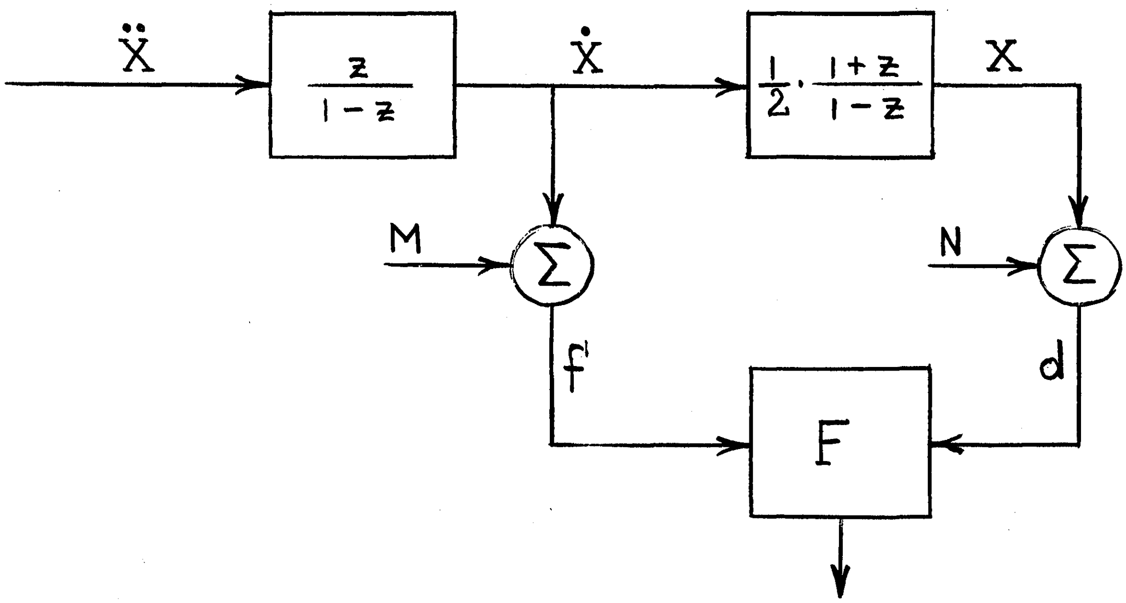

Figure 2

As a starting point, we assume the form of the probability distribution of the samples of the function and the derivative, given all the past and present data. Let L_n represent all data up to and including the n-th sample. In particular, let

$$W_1\left(X_{n-1}, V_{n-1}/L_{n-1}\right) = \frac{1}{K} e^{-A\left(X_{n-1}+BV_{n-1}+C\right)^2 - D\left(V_{n-1}+E\right)^2} \tag{6}$$

where A, B, C, D and E are arbitrary constants, K is a normalizing factor, and V_n denotes $\dot{X}_n$. By changing the variables to express the distribution about the mean, Eq. (6) reduces to

$$W_1'\left(X_{n-1}, V_{n-1}/L_{n-1}\right) = \frac{1}{K} e^{-A\left(X_{n-1}+BV_{n-1}\right)^2 - DV_{n-1}^2} \tag{7}$$

When all transients have died out and when the system has reached steady state, the distribution described by Eq. (7) will become invariant and independent of n. In matrix notation, this distribution must attain the form

$$W_1'(x_n, v_n/L_n) = \frac{\sqrt{|\Psi|}}{2\pi} e^{-\frac{1}{2}\underline{x_n\, v_n}\,[\Psi]\begin{bmatrix} x_n \\ v_n \end{bmatrix}} \tag{8}$$

regardless of the value of n. Here Ψ is the inverse of the correlation matrix Φ.

$$\Phi = \Psi^{-1} = \begin{bmatrix} \phi_{xx} & \phi_{xv} \\ \phi_{vx} & \phi_{vv} \end{bmatrix}$$

In the notation of Eq. (7),

$$\Phi = \begin{bmatrix} 2A & 2AB \\ 2AB & 2(AB^2+D) \end{bmatrix}^{-1} = \frac{1}{2AD}\begin{bmatrix} AB^2+D & -AB \\ -AB & A \end{bmatrix} \tag{9}$$

Therefore, the values of the correlation coefficients

$$\left\{ \begin{aligned} \phi_{xx} &= \frac{AB^2+D}{2AD} \\ \phi_{vv} &= \frac{1}{2D} \end{aligned} \right. \qquad \phi_{xv} = -\frac{B}{2D} \tag{10}$$

must remain the same for all values of n.

By computing from the distribution expressed by Eq. (6) the form of distribution corresponding to the next higher value of n, and by comparing these expressions and equating respective correlation coefficients, it is possible to obtain and solve a set of simultaneous algebraic equations in A, B, and D.

The first step in this procedure is to form:

$$W(x_n, V_m/L_{n-1}) =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(x_n, V_n/x_{n-1}, V_{n-1}, a_{n-1}, L_{n-1}) W(x_{n-1}, V_{n-1}, a_{n-1}/L_{n-1})\, dx_{n-1}\, dV_{n-1}\, da_{n-1}$$

Here, V_n and a_n denote $\dot{X}_n$ and $\ddot{X}_n$ respectively. Using Eqs. (1), (2), (3), and (6), and integrating, the above expression can be reduced to

$$W(x_n, V_n/L_{n-1}) = \frac{1}{K'} e^{\frac{q^2}{4p} - r} \tag{11}$$

where

$$\begin{cases} p = A\left(\frac{1}{2}-B\right)^2 + D + \frac{1}{2\alpha^2} \\ q = 2\left\{A\left(\frac{1}{2}-B\right)x_n + \left[A\left(\frac{1}{2}-B\right)(B-1) - D\right]V_n + AC\left(\frac{1}{2}-B\right) - DE\right\} \\ r = Ax_n^2 + \left[A(B-1)^2 + D\right]V_m^2 + 2A(B-1)x_nV_n + 2ACx_n + \\ \quad + 2\left[AC(B-1) + DE\right]V_m + AC^2 + DE^2 \end{cases}$$

From Eqs. (4) and (5):

$$W(d_n/x_n, L_{n-1}) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(d_n - x_n)^2}{2\sigma^2}} \tag{12}$$

and

$$W(f_n/V_n, L_{n-1}) = \frac{1}{\sqrt{2\pi}\,\mu} e^{-\frac{(f_n - V_n)^2}{2\mu^2}} \tag{13}$$

Since the noise in the samples of the function is assumed to be independent of the noise in the samples of the derivative,

$$W(d_n, f_n/x_n, V_n, L_{n-1}) = \frac{1}{2\pi\sigma\mu} e^{-\frac{(d_n - x_n)^2}{2\sigma^2} - \frac{(f_n - V_n)^2}{2\mu^2}} \tag{14}$$

But,

$$W(x_n, v_n/L_n) = W(x_n, v_n/d_n, f_n, L_{n-1}) =$$

$$= \frac{W(x_n, v_n, d_n, f_n/L_{n-1})}{W(d_n, f_n/L_{n-1})} = \tag{15}$$

$$= \frac{W(d_n, f_n/x_n, v_n, L_{n-1})W(x_n, v_n/L_{n-1})}{\iint W(d_n, f_n/x_n, v_n, L_{n-1})W(x_n, v_n/L_{n-1})\, dx_n\, dv_n}$$

Substituting from Eqs. (11) and (14),

$$W(x_n, v_n/L_n) = \frac{1}{K''} e^{-U_n} \tag{16}$$

where

$$U_n = \frac{(d_n - x_n)^2}{2\sigma^2} + \frac{(f_n - v_n)^2}{2\mu^2} - \frac{q^2}{4p} + r + S$$

and S is a function of f_n and d_n only.

From Eq. (6) the corresponding expression for the previous sample is:

$$U_{n-1} = A(x_{n-1} + Bv_{n-1} + C)^2 + D(v_{n-1} + E)^2$$

Equating the coefficients of second degree terms in X and V and solving the resulting set of three simultaneous equations, one obtains:

$$\left\{ \begin{aligned} A &= \frac{\lambda}{2\sigma^2} + \sqrt{\left(\frac{\lambda}{2\sigma^2}\right)^2 + \frac{1}{4\sigma^2\mu^2}} \\ B &= \frac{1}{2} - \lambda \\ D &= \frac{A\left(\frac{1}{2} - \lambda\right)\lambda^2 + \frac{1}{2\alpha^2}}{\lambda - 1} \end{aligned} \right. \tag{17}$$

where
$$\lambda = \frac{1}{2}\left(\frac{\gamma+\frac{1}{2}}{\gamma+1}\right) + \sqrt{\frac{1}{4}\left(\frac{\gamma+\frac{1}{2}}{\gamma+1}\right)^2 + \frac{\gamma k_2}{\gamma+1}}$$

with
$$\gamma = \sqrt{\frac{k_1}{4k_2^2 + k_2}}, \quad k_1 = \frac{\sigma^2}{\alpha^2}, \quad k_2 = \frac{\mu^2}{\alpha^2}.$$

By using Eqs. (10) and (17), it is now possible to compute the correlation coefficients as functions of α, σ, and μ. The normalized non-dimensional ratios $\frac{\phi_{xx}}{2\alpha^2}$ and $\frac{\phi_{vv}}{2\alpha^2}$ are functions of the parameters k_1 and k_2 only. They are plotted in Figs. 1A and 2A as functions of k_2 for two values of k_1. For large values of k_2 these curves approach asymptotically the normalized variance of error in the smoothed samples of the function and its derivative in the case when the measured samples of the derivative are not available.

It is of interest to investigate also the accuracy of the samples of the function and the derivative predicted one sampling interval into the future. Eqs. (2) and (3) can be written

$$\begin{cases} x_{n+1} = x_n + v_n + \frac{1}{2} a_n \\ v_{n+1} = v_n + a_n \end{cases} \tag{18}$$

Since the mean of the acceleration is zero,

$$\begin{cases} \overline{x_{n+1}} = \overline{x_n} + \overline{v_n} \\ \overline{v_{n+1}} = \overline{v_n} \end{cases} \tag{19}$$

Therefore, the expressions for the variance of predicted values are

$$\begin{aligned} \sigma_p^2(x) &= \overline{\left(x_{n+1} - \overline{x_{n+1}}\right)^2} = \overline{\left(x_m - \overline{x_n} + v_n - \overline{v_n}\right)^2} + \frac{1}{4}\overline{a_n^2} \\ \sigma_p^2(v) &= \overline{\left(v_{n+1} - \overline{v_{n+1}}\right)^2} = \overline{\left(v_n - \overline{v_n}\right)^2} + \overline{a_n^2} \end{aligned} \tag{20}$$

since a_n is independent of X_n and V_n.

Eqs. (20) are equivalent to

$$\begin{cases} \sigma_P^2(x) = \phi_{xx} + \phi_{vv} + 2\phi_{xv} + \frac{1}{4}\alpha^2 \\ \sigma_P^2(v) = \phi_{vv} + \alpha^2 \end{cases} \tag{21}$$

The normalized, non-dimensional ratios $\dfrac{\sigma_P^2(x)}{2\alpha^2}$ and $\dfrac{\sigma_P^2(v)}{2\alpha^2}$ are plotted in Figs. 3A and 4A as functions of k_2 for two different values of k_1. For large values of k_2 these curves approach asymptotically the normalized variance of error in the predicted samples of the function and its derivative in the case when only the measured samples of the function are available.

Fig. 5A shows the functional dependence of the variance of error of the smoothed and predicted samples of the function on the parameter k_1 for one particular value of k_2.

To investigate the effect of acceleration on the accuracy of prediction, a plot of the variance of error in the predicted samples of the function against the variance of the probability density distribution of the second derivative is shown in Fig. 6A. The family of curves corresponds to several representative values of the standard deviation of the error in the measured samples of the first derivative, while the standard deviation of the error in the measured samples of the function is kept constant. As can be seen from Fig. 6A, the less accurately we measure the first derivative, the more rapidly does the accuracy of prediction deteriorate with increasing variance of the second derivative.

The results of this analysis describe statistically the errors in the output of an optimum linear smoothing filter in the minimum mean square error sense. This filter is realizable, and the associated smoothing equations can be obtained from the mean of the distribution described by Eq. (16). They are

$$
\left\{
\begin{aligned}
\overline{X_n} &= \overline{X_{n-1}} + \frac{\phi_{xx}}{\sigma^2}\left(d_n - \overline{X_{n-1}}\right) + \frac{\phi_{xv}}{\mu^2} \cdot f_n + \\
&\qquad + \frac{2\,\phi_{xx}^2 + \phi_{xv}\left(\phi_{xx} - 2\sigma^2\right)}{\sigma^2\left(\phi_{vv} + 2\,\phi_{xv}\right)} \cdot \overline{V_{n-1}} \\
\overline{V_n} &= \frac{\phi_{vv}}{\mu^2} f_n + \frac{\phi_{xv}}{\sigma^2}\left(d_n - \overline{X_{n-1}}\right) + \\
&\qquad + \frac{\sigma^2 \phi_{vv} + \phi_{xv}\left(2\,\phi_{xx} + \phi_{xv} - 2\sigma^2\right)}{\sigma^2\left(\phi_{vv} + 2\,\phi_{xv}\right)}
\end{aligned}
\right.
$$

Signed B. Smulowicz
Bronislaw Smulowicz

Approved William K. Linvill
William K. Linvill W.I.W.

BS/mrs
cc: Group 61
6782 Distribution List

Figures Attached: Fig. 1A A-58459
Fig. 2A A-58460
Fig. 3A A-58461
Fig. 4A A-58448
Fig. 5A A-58449
Fig. 6A A-58450

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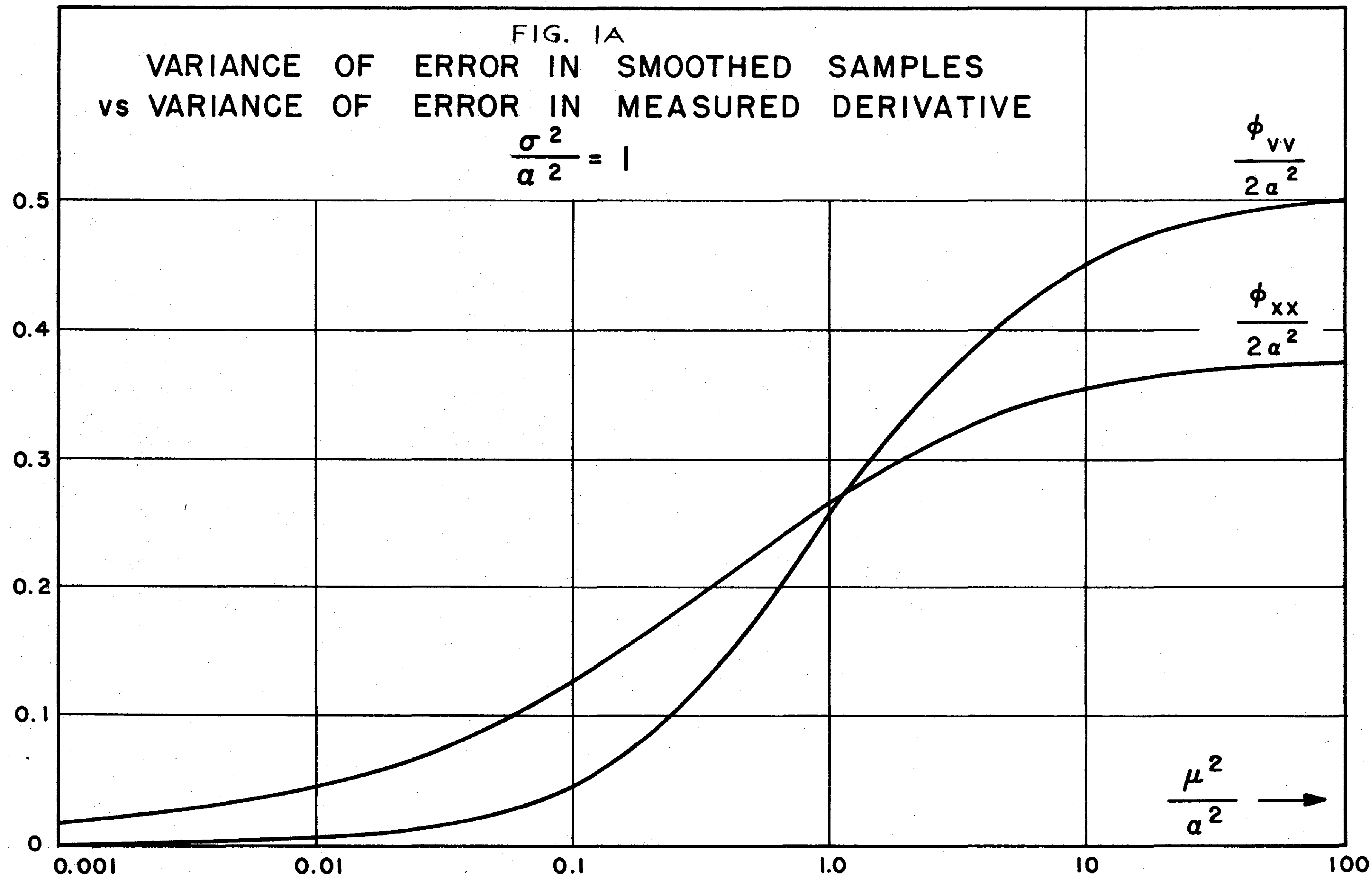
FIG. 1A
VARIANCE OF ERROR IN SMOOTHED SAMPLES
vs VARIANCE OF ERROR IN MEASURED DERIVATIVE
$\frac{\sigma^2}{\alpha^2} = 1$
$\frac{\phi_{vv}}{2\alpha^2}$
$\frac{\phi_{xx}}{2\alpha^2}$
0.5
0.4
0.3
0.2
0.1
0
0.001
0.01
0.1
1.0
10
100
$\frac{\mu^2}{\alpha^2}$

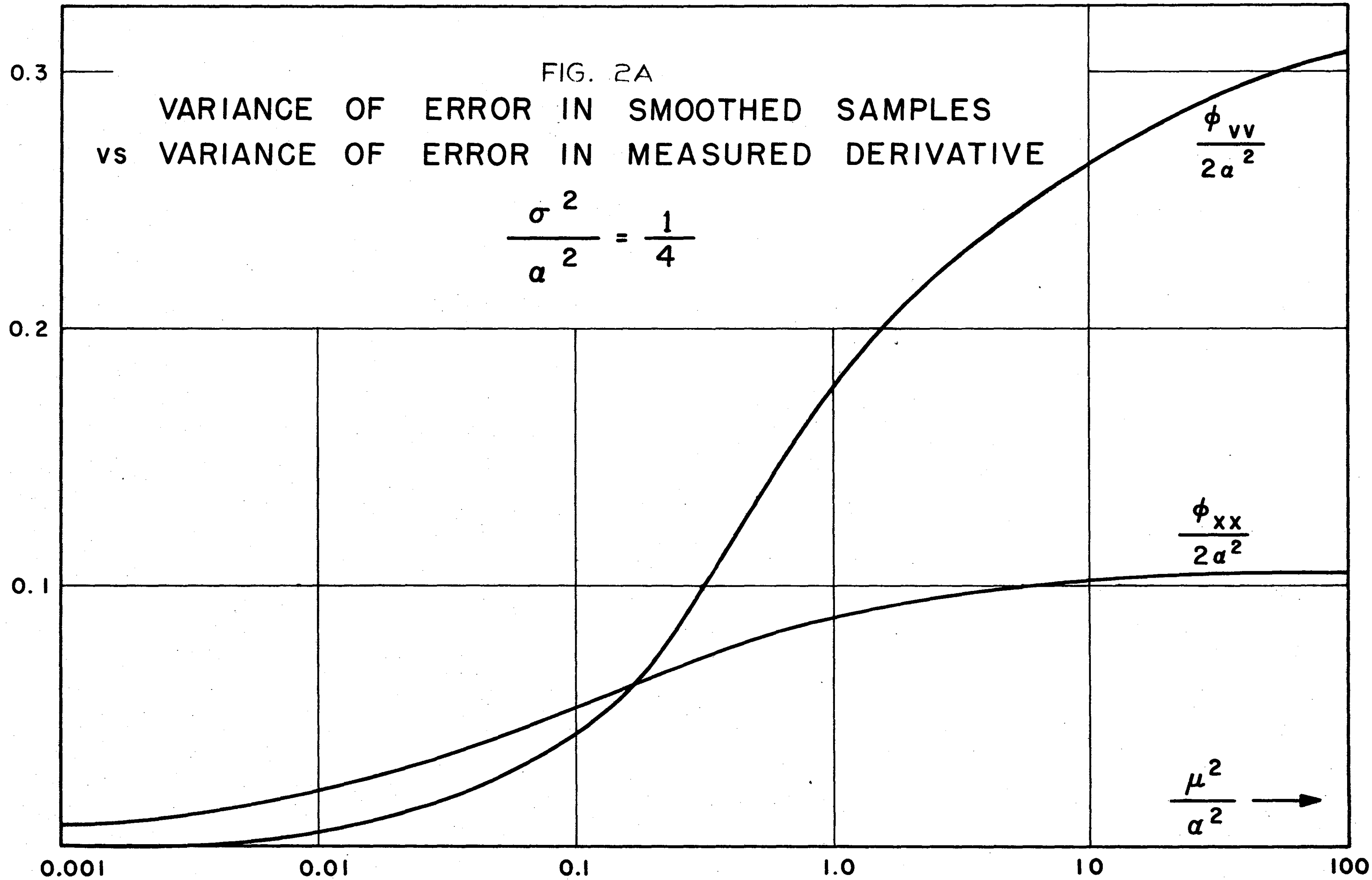
FIG. 2A
VARIANCE OF ERROR IN SMOOTHED SAMPLES
vs VARIANCE OF ERROR IN MEASURED DERIVATIVE
$\frac{\sigma^2}{a^2} = \frac{1}{4}$
$\frac{\phi_{vv}}{2a^2}$
$\frac{\phi_{xx}}{2a^2}$
$\frac{\mu^2}{a^2}$
0.3
0.2
0.1
0.001
0.01
0.1
1.0
10
100

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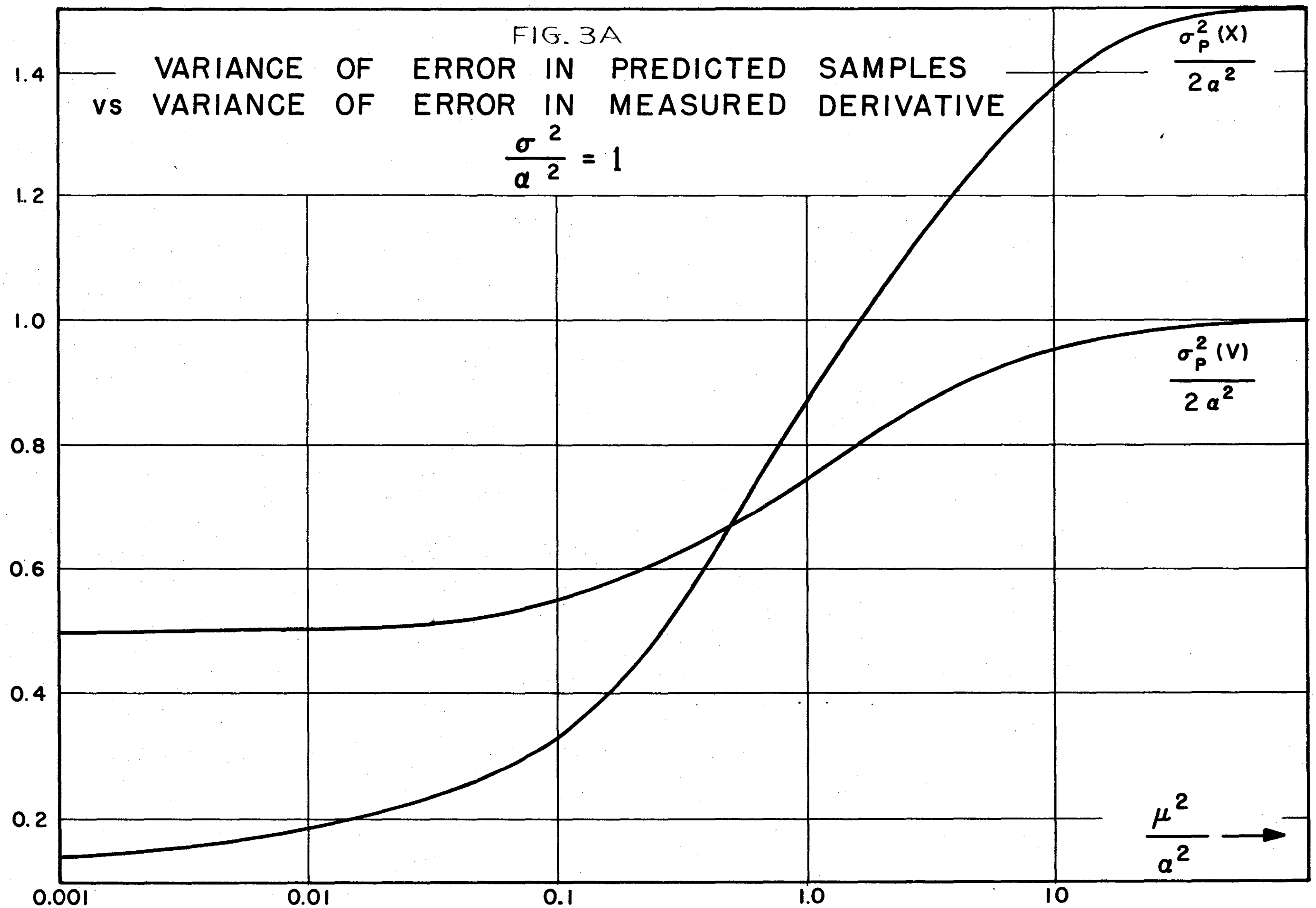

FIG. 3A
VARIANCE OF ERROR IN PREDICTED SAMPLES
vs VARIANCE OF ERROR IN MEASURED DERIVATIVE
$\frac{\sigma^2}{a^2} = 1$
$\frac{\sigma_P^2(X)}{2a^2}$
$\frac{\sigma_P^2(V)}{2a^2}$
$\frac{\mu^2}{a^2}$
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.001
0.01
0.1
1.0
10

A-58448

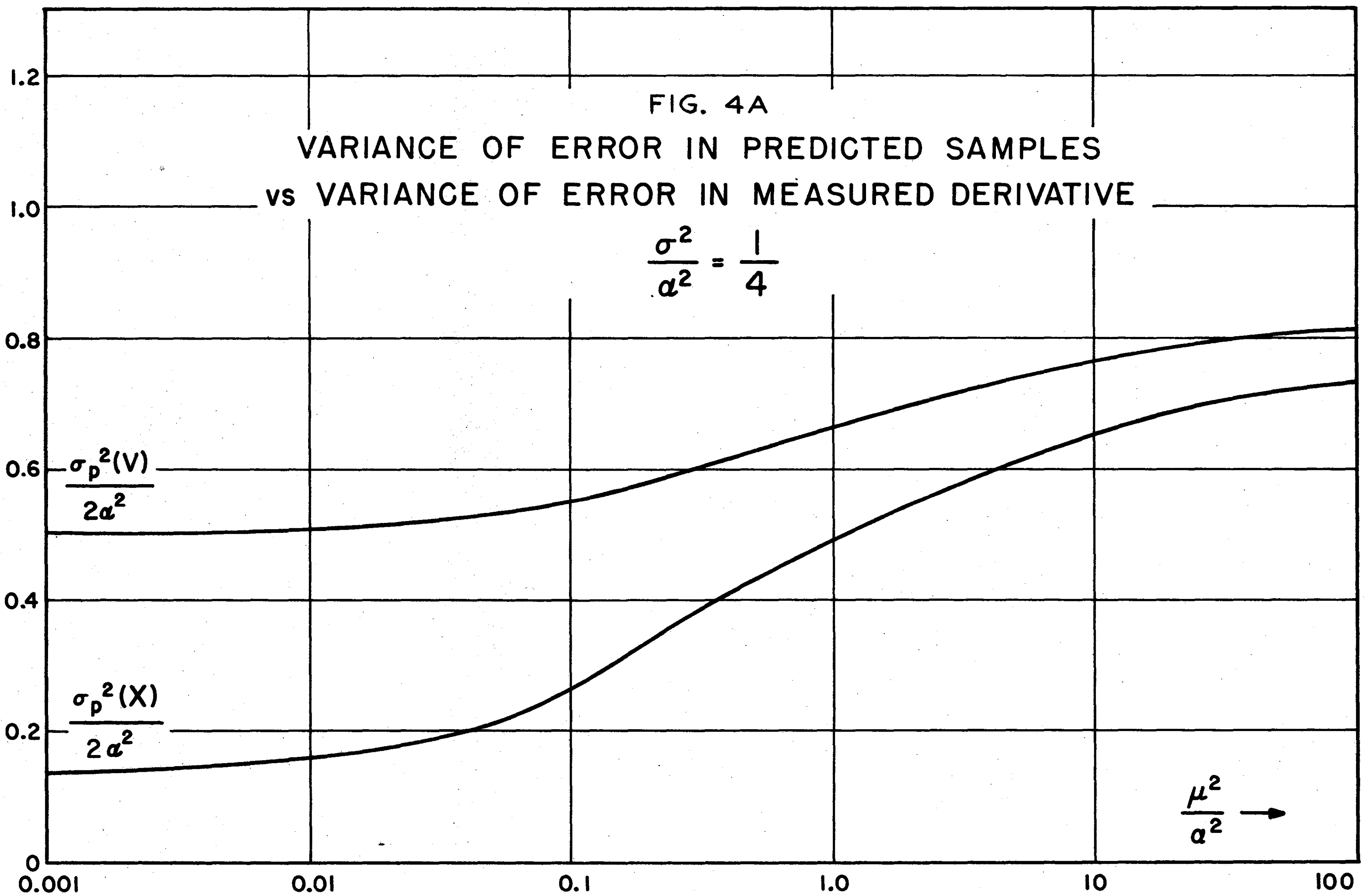
FIG. 4A
VARIANCE OF ERROR IN PREDICTED SAMPLES
vs VARIANCE OF ERROR IN MEASURED DERIVATIVE
$\frac{\sigma^2}{a^2} = \frac{1}{4}$
$\frac{\sigma_p^2(V)}{2a^2}$
$\frac{\sigma_p^2(X)}{2a^2}$
$\frac{\mu^2}{a^2}$
0
0.2
0.4
0.6
0.8
1.0
1.2
0.001
0.01
0.1
1.0
10
100

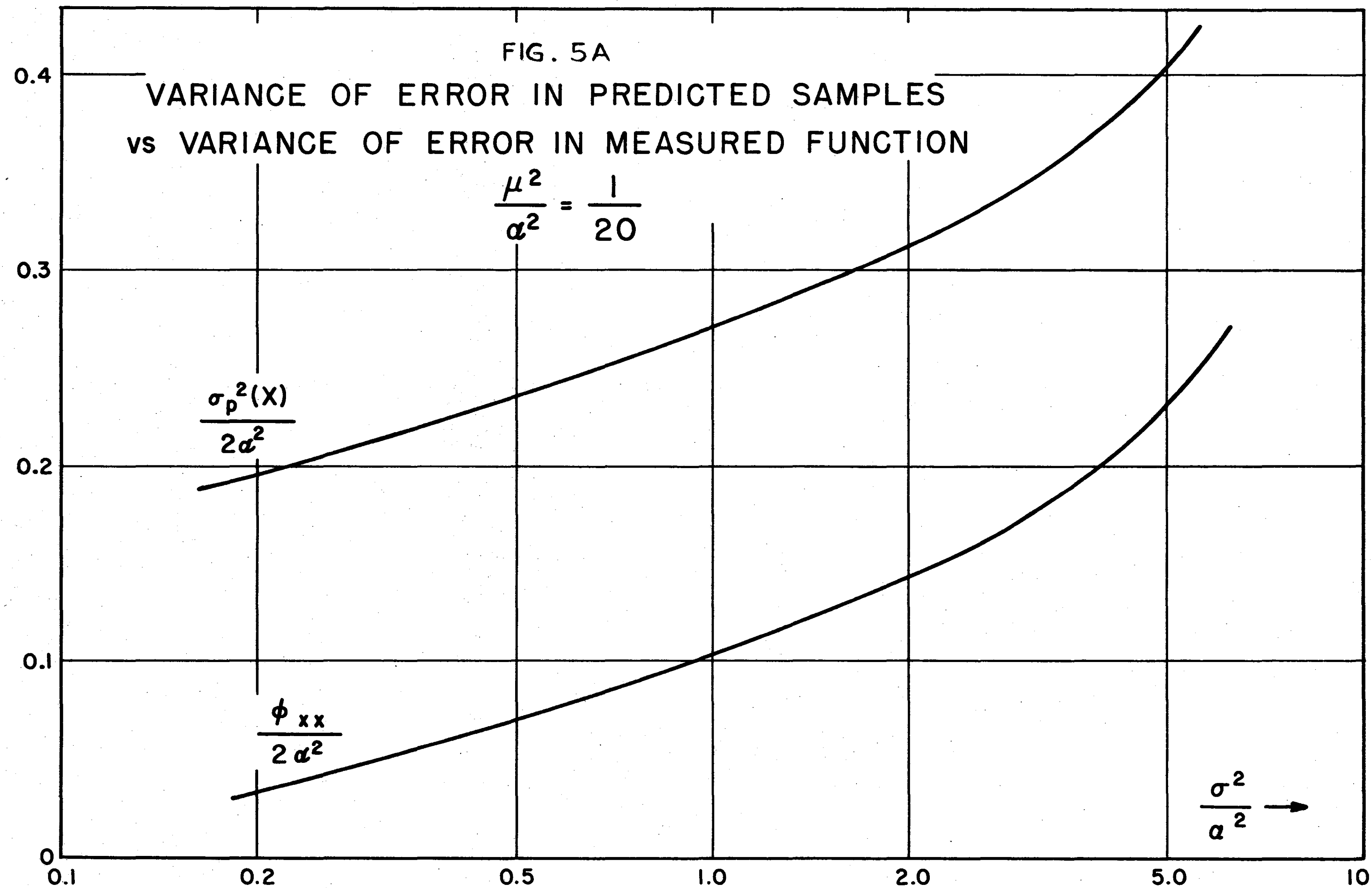
FIG. 5A
VARIANCE OF ERROR IN PREDICTED SAMPLES
vs VARIANCE OF ERROR IN MEASURED FUNCTION
$\frac{\mu^2}{\alpha^2} = \frac{1}{20}$
$\frac{\sigma_p^2(X)}{2\alpha^2}$
$\frac{\phi_{xx}}{2\alpha^2}$
$\frac{\sigma^2}{\alpha^2}$
0
0.1
0.2
0.3
0.4
0.1
0.2
0.5
1.0
2.0
5.0
10

A-58450

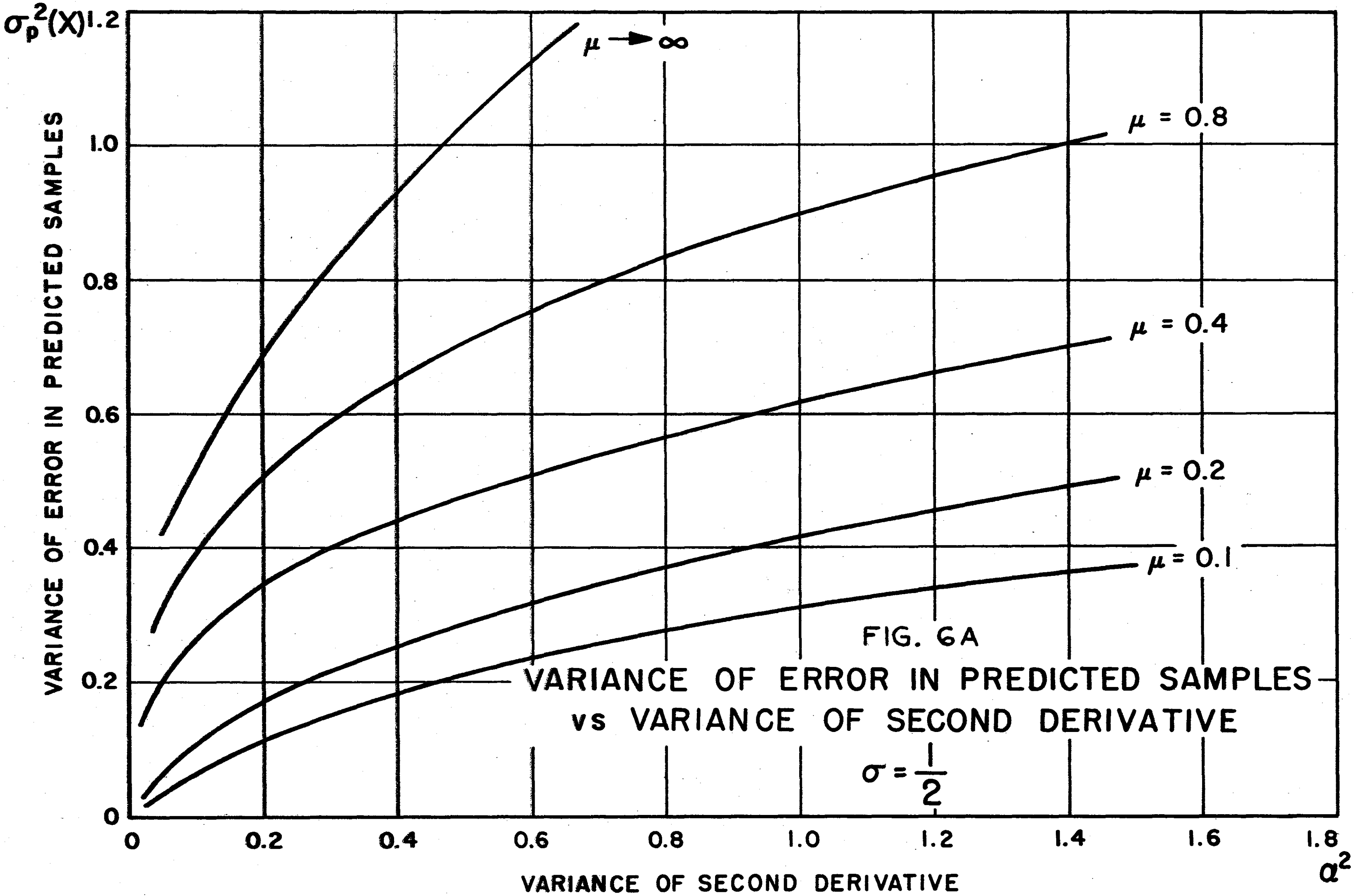

FIG. 6A

VARIANCE OF ERROR IN PREDICTED SAMPLES vs VARIANCE OF SECOND DERIVATIVE

$\sigma = \frac{1}{2}$